Nigerian Journal of Theoretical and Environmental Physics (NJTEP)
ISSN Print: 3026-9601
DOI: https://doi.org/10.62292/njtep.v2i2.2024.45
Volume 2(2), June 2024

Complete Synchronization, Anti-Synchronization and Hybrid Synchronization of Non-Identical Chaotic Financial Systems

***[1]Oyeleke, K. S., [2]Ojo, K. S., [3]Adeniji, A. E. and [2]Olusola, O. I.**

[1]Department of Physical Sciences, Lagos State University of Science and Technology, Ikorodu, Lagos, Nigeria.
[2]Department of Physics, University of Lagos, Lagos, Nigeria.
[3]Department of Physical Sciences, Bells University of Technology, Ota, Nigeria.

*Corresponding author's email: ksoyeleke@gmail.com Phone: +2348037850388

ABSTRACT

Chaos theory, being a branch of mathematics that deals with disordered or random-seeming mathematical systems, is receiving more attention in market-related circumstances as financial markets become more unstable and the level of unpredictability becomes more prevalent. Understanding the potential of chaos theory, its limitations, and its relationship to traditional economic theories is essential for anyone working in the finance sector. Chaos theory is ideally suited for comprehending the financial market, which is subject to both internal and external influences due to its high degree of instability and growing randomness. This work examines the complete synchronization, anti-synchronization and hybrid synchronization of two non-identical financial systems. The nonlinear active controllers are designed, and the error dynamics stability for each phenomenon is accomplished by two theoretical approaches - linear system theory and Lyapunov second method. Controllers are designed by using the relevant variables of drive and response systems in such a way that the error variables are stable. The controllers, when activated, enable the drive and response state variables to achieve identical dynamics despite starting from different initial conditions. Numerical simulations are performed using the ODE45 algorithm embedded in MATLAB software package to show the feasibility and effectiveness of the designed controllers.

Keywords:
Chaos theory,
Financial system,
Complete synchronization,
Anti-synchronization,
Hybrid synchronization.

INTRODUCTION

In various branches of physics, many models are suggested to study real-world systems and their dynamical behaviours are studied via the Lagrangian or the Hamiltonian approach (Baleanu *et al.*, 2020). These dynamical behaviours are being mod-elled as dynamical systems described by differential equations (Wen *et al.*, 2017); accordingly, the parameters' effects on such dynamical systems are important (Kilikevic-ius *et al.*, 2015). Chaos theory has found widespread appli-cation in finance because of the unpredict-able behaviour of market participants (Hsieh, 1991). For example, when examining stockholders' equity or cash flow at the end of the year, both are indexed by profit and loss; different values of profit and loss are associated with the same values of stockholders' equity and cash flow at the end of the year, suggesting the lack of a unique solution to explain this relationship and the difficulty of making predictions. Consequently, at the end of the year, stockholders' equity and cash flow are thus greatly affected by even slight variations in profit or loss. These discoveries are suggestive and indicative, introducing the language of chaos to the management view of financial statements (Juarez, 2015).

Over the past few decades, the study of chaos has considerably impacted the fundamentals of science and engineering. One of the most interesting discoveries in this field is the discovery of chaotic synchronization, a notion that was first suggested by Fujisaka and Yamada (1983). Synchronizing two chaotic systems that are identical or non-identical and have distinct initial conditions is one of the significant advances in the study of nonlinear science. A significant article on the synchronization of chaotic systems employing a drive-response arrangement was later presented by Pecora and Carroll (1990). For the response system's outputs to synchronize with the drive system's trajectory, the drive system's outputs are used to regulate the response system.

Since the discovery of the synchronization of chaotic systems, the study of the synchronization of chaotic and hyperchaotic systems arising from various initial conditions has attracted a lot of attention. It has sparked nearly two decades of vigorous research and has been hailed as a major advancement in chaotic dynamics (Freeman,1992), information processing, and secure communication (Gabriel and Hilda, 1995; Pikovsky *et al.*, 2002; Bowang *et al.*, 2007; Eisencraft *et al.*, 2012; Ren *et al.*, 2013; Aguilar_Lopez *et al.*, 2014; Filali and Pierre, 2014; Olusola *et al.*, 2020) because of its crucial and interdisciplinary applications in a variety of scientific and technological studies.

Complex chaotic systems are susceptible to small perturbations, as demonstrated by Lorenz, and these can upset a system and cause it to deviate significantly from its equilibrium (Wu *et al.*, 2007; Rameika, 2007). Two fundamental feedback and causal loops in market system dynamics affect different facets of the stock market. A positive feedback loop reinforces itself. For instance, a positive influence in one variable raises the other, which raises the first variable as well. As a result, the system experiences exponential growth, which pushes it out of equilibrium and ultimately causes the system to collapse. On the other hand, a negative feedback loop produces a similar result, with the system reacting to a change in the opposite way. High uncertainty periods could result from factors other than system dynamics. Market volatility can also be brought on by external forces like earthquakes, floods, or natural disasters, as well as by abrupt declines in a particular stock (Bourdeau-Brien and Kryzanowski 2017).

Some properties of systems that have historically proven challenging to adequately model have been explained by the contentious and complex idea of chaos. The financial markets are one example of this, and they also have the advantage of having a wealth of historical data. The ability of apparently sound financial markets to experience abrupt shocks and crashes is an intriguing financial phenomenon that chaos theory can assist in explaining (Lu, 2020). The argument made by proponents of chaos theory is that price changes last for stocks, bonds, and other securities; therefore, periods of low price volatility could not be a reliable indicator of the state of the market. Investors are left in the dark regarding the timing of crashes when the price is used as a lagging indicator.

According to the fractal market theory, price movements during periods of market turbulence could follow a fractal pattern as opposed to a random walk (Metescu, 2022). Put differently, it is possible for movements that take place on a tiny time scale to repeat on a greater scale. Naturally, this aligns with the experiences of most investors who have gone through financial meltdowns and black swan events. Some people appear to be able to anticipate market downturns, but they frequently look much beyond price data to identify structural flaws that the majority of the market has missed. The primary caution associated with chaos theory is its frequent application as a means of devaluing investments (Klioutchnikov *et al.*, 2017). Long-term market consistency is higher than short-term market volatility, which is nearly impossible to anticipate.

The Fractal Market Hypothesis in finance forecasts stock market fluctuations by utilizing aspects of chaos theory (Arashi and Rounaghi, 2022). The efficient market hypothesis, which postulates that prices move in a random walk, is expanded upon by this theory. According to the fractal market theory, price movements might exhibit similar behaviour when observed over several time horizons under periods of high uncertainty. This could be applied to technical analysis, where future price movements can be predicted using recursive or repeating patterns. Consequently, there is a compelling need to synchronize and control the chaotic behaviour of financial systems by designing appropriate controller. Thus, this present work.

Systems Description and Equilibrium Analysis

Two different financial systems are considered in this work and their stabilities are also investigated.

Systems Description

A chaotic financial system, which has been investigated by Liao *et al.*, (2020), Dousseh *et al.*, (2022) and Tusset *et al.*, (2023), is describable by the following autonomous differential equations:

$$\begin{cases} \dot{x}_1 = fx_3 + x_2x_1 - ax_1 \\ \dot{x}_2 = -bx_2^2 - hx_1^2 + \tau \\ \dot{x}_3 = -cx_3 - gx_1 - dx_2, \end{cases} \qquad (1)$$

where a, b, c, d, τ, f, g and h are parameters that determine the behaviour of the system. The parameter values are $a = 0.3, b = 0.02, c = 1, d = 0.05, \tau = 1, f = 1.2, g = 1,$ and $h = 0.1.$ x_1, x_2, x_3 are state variables which evolve with time. x_1 represents the interest rate, x_2 represents the investment demand and x_3 represents the price index. The variation in the interest rate is proportional to the price index, and the investment demand influences the rate interest (Liao *et al.*, 2020; Dousseh *et al.*, 2022; Tusset *et al.*, 2023). With the choice of the parameter values and initial condition (0,0,0); the phase diagram of Equation (1) is as shown in Figure 1.

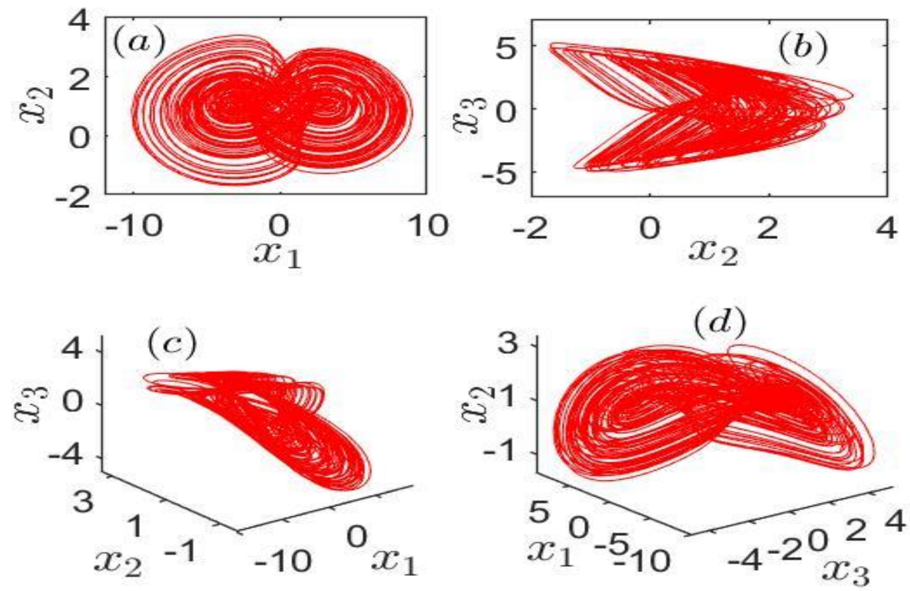

Figure 1: Phase portrait of Equation (1) with the following parameters $a = 0.3,\ b = 0.02, c = 1, d = 0.05, \tau = 1, f = 1.2, g = 1, h = 0.1$

Another chaotic financial system by Xiao-Dan *et al.*, (2013) is describable by following autonomous differential equations:

$$\begin{cases} \dot{y}_1 = p(y_2 - y_1) + y_2 y_3 \\ \dot{y}_2 = q y_1 - y_2 - y_1 y_3 \\ \dot{y}_3 = y_1 y_2 - r y_3, \end{cases} \qquad (2)$$

where y_1, y_2 and y_3 are also state variables that evolve with time and denote the occurrence value risk, the analysis value risk and the control value risk in the current market, respectively. Parameters $p, q$and r determine the system's behaviour. With the parameter values$p = 10, q = 28, r = \frac{8}{3}$; and initial condition (1,1,1); the phase diagram of Equation (2) is as shown in Figure 2.

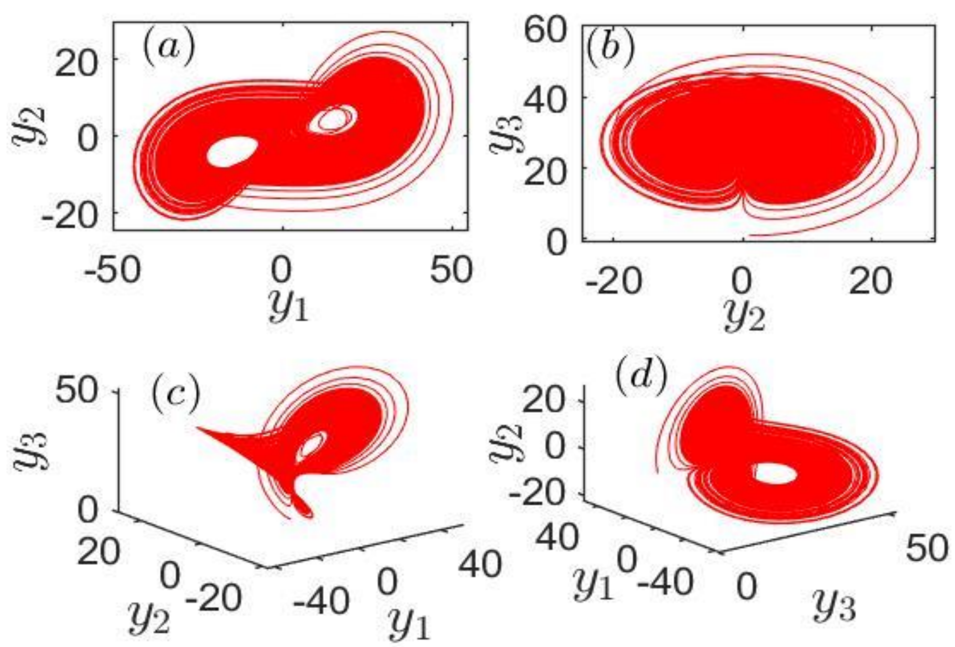

Figure 2: Phase portrait of Equation (1) with the following parameters$p = 10, q = 28, r = \frac{8}{3}$.

The divergence of Equation (1) is in the form:

$$\begin{cases} \Delta V = \frac{d\dot{x}_1}{dx_1} + \frac{d\dot{x}_2}{dx_2} + \frac{d\dot{x}_3}{dx_3} \\ \Delta V = -a - b - c. \end{cases} \quad (3)$$

Obviously, for the parameter values considered, $\Delta V < 0$. Consequently, Equation (1) is dissipative (Gan, 2021).

Also, the divergence of Equation (2) is in the form:

$$\begin{cases} \Delta V = \frac{d\dot{y}_1}{dy_1} + \frac{d\dot{y}_2}{dy_2} + \frac{d\dot{y}_3}{dy_3} \\ \Delta V = -p - 1 - r. \end{cases} \quad (4)$$

For the parameter values considered, Equation (2) is also dissipative.

Equilibrium Analysis

The equilibria of Equation (1) can be obtained by solving:

$$\begin{cases} fx_3 + x_2x_1 - ax_1 = 0 \\ -bx_2^2 - hx_1^2 + \tau = 0 \\ -cx_3 - gx_1 - dx_2 = 0, \end{cases} \quad (5)$$

Linearizing the system around the equilibrium pointE_0, it yields the following Jacobian matrix:

$$J = \begin{vmatrix} -(a - x_2) & x_1 & f \\ -h & -b & 0 \\ -g & -d & -c \end{vmatrix}. \quad (6)$$

The eigenvalues can be determined by solving the following cubic equation:

$$-\lambda^3 + (a + b + c - x_2)\lambda^2 + (ab + ac + bc + fg - bx_2 - cx_2 + hx_1)\lambda + abc + bfg - dfh + chx_1 - bcx_2 = 0. \quad (7)$$

For the centrally located equilibrium point $E_0(0,0,0)$and parameters values $a = 0.3, b = 0.02, c = 1, d = 0.05, f = 1.2, g = 1$ and $h = 0.1$, Equation (6) yields eigenvalues of: $\lambda_{1,2} = -0.6520 \pm 1.0393i$, $\lambda_3 = -0.0159$. All the eigenvalues have negative real parts, therefore the equilibrium point of Equation (1) is a stable equilibrium point (Woolf, 2009).

Also, the equilibria of Equation (2) can be obtained by solving:

$$\begin{cases} p(y_2 - y_1) + y_2y_3 = 0 \\ qy_1 - y_2 - y_1y_3 = 0 \\ y_1y_2 - ry_3 = 0, \end{cases} \quad (8)$$

and linearizing the system around E_0, it yields the Jacobian matrix:

$$J = \begin{vmatrix} -p & (p + y_3) & y_2 \\ (q - y_3) & -1 & -y_1 \\ y_2 & y_1 & -r \end{vmatrix}. \quad (9)$$

The eigenvalues can be determined by solving the following cubic equation:

$$\lambda^3 + (1 + p + r)\lambda^2 + (p(1 + r - q + y_3) - qy_3 + r + y_1^2 - y_2^2 + y_3^2)\lambda + p(r - qr + y_1^2 + y_1y_2 + ry_3) + ry_3^2 - qry_3 - qy_1y_2 - y_2^2 + 2y_1y_2y_3 = 0. \quad (10)$$

For the centrally located equilibrium point $E_0(0,0,0)$and parameters values$p = 10, q = 28$ and $r = 8/3$, Equation (9) yields eigenvalues of: $\lambda_1 = -22.8277, \lambda_2 = 11.8277$, $\lambda_3 = -2.6667$. Looking at these eigenvalues, it is clear that the equilibrium point of Equation (2) is an unstable saddle point (Woolf, 2009).

Synchronization of the Financial Systems

The complete synchronization, the anti-synchronization and the hybrid synchronization of the financial systems are investigated in this section, using the active control method. Equation (1) is taken as the master system while Equation (2) is taken as the slave system which can be expressed in the form:

$$\begin{cases} \dot{y}_1 = p(y_2 - y_1) + y_2y_3 + u_1 \\ \dot{y}_2 = qy_1 - y_2 - y_1y_3 + u_2 \\ \dot{y}_3 = y_1y_2 - ry_3 + u_3, \end{cases} \quad (11)$$

where u_1, u_2 and u_3 are the active controllers to be designed.

Complete Synchronization

The complete synchronization error is defined by

$$e_i = y_i - x_i, (i = 1,2,3). \quad (12)$$

Then, the complete synchronization error of Equation (1) and Equation (11) is obtained as

$$\begin{cases} \dot{e}_1 = p(e_2 - e_1) + e_2e_3 + (a - p - x_2)x_1 + \\ (p + e_3 + x_3)x_2 + (e_2 - f)x_3 + u_1, \\ \dot{e}_2 = qe_1 - e_2 - e_1e_3 - (e_3 - q - hx_1)x_1 - \\ (1 - bx_2)x_2 - (e_1 + x_1)x_3 - \tau + u_2, \\ \dot{e}_3 = e_1e_2 - re_3 + (e_2 + g + x_2)x_1 + \\ (e_1 + d)x_2 - (r - c)x_3 + u_3. \end{cases} \quad (13)$$

The nonlinear active control law for u_i, $(i = 1,2,3)$ should be constructed in such a manner that the error dynamics Equation (13) is globally stable.

Choosing the control function $u = [u_1, u_2, u_3]^T$ as

$$\begin{cases} u_1 = -e_2e_3 - (a - p - x_2)x_1 - \\ (p + e_3 + x_3)x_2 + (f - e_2)x_3 + v_1, \\ u_2 = e_1e_3 + (e_3 - q - hx_1)x_1 + \\ (1 - bx_2)x_2 + (e_1 + x_1)x_3 + \tau + v_2, \\ u_3 = -e_1e_2 - (e_2 + g + x_2)x_1 - \\ (e_1 + d)x_2 + (r - c)x_3 + v_3. \end{cases} \quad (14)$$

where v_1, v_2 and v_3 are the linear control input chosen such that Equation (13) becomes stable.

Substituting Equation (14) into Equation (13) yields

$$\begin{cases} \dot{e}_1 = p(e_2 - e_1) + v_1, \\ \dot{e}_2 = qe_1 - e_2 + v_2, \\ \dot{e}_3 = -re_3 + v_3. \end{cases} \quad (15)$$

Let us consider

$$v = [v_1, v_2, v_3]^T = M[e_1, e_2, e_3], \quad (16)$$

where $M$is a 3 by 3 constant matrix. For stability, the matrix M is selected in such a way that all of its eigenvalues are with negative real parts. Consider the following choice of matrix Mas

$$M = \begin{vmatrix} p-1 & -p & 0 \\ -q & 0 & 0 \\ 0 & 0 & r-1 \end{vmatrix}; \quad (17)$$

then,

$$|v_1, v_2, v_3| = \begin{vmatrix} p-1 & -p & 0 \\ -q & 0 & 0 \\ 0 & 0 & r-1 \end{vmatrix} \begin{vmatrix} e_1 \\ e_2 \\ e_3 \end{vmatrix}, \quad (18)$$

leads to

$$\begin{cases} v_1 = (p-1)e_1 - pe_2, \\ v_2 = qe_1, \\ v_3 = (r-1)e_3. \end{cases} \quad (19)$$

Hence, Equation (1) and Equation (11) will achieve complete synchronization when the nonlinear active control is chosen as

$$\begin{cases} u_1 = (p-1)e_1 - pe_2 - e_2e_3 - (a-p- \\ x_2)x_1 - (p+e_3+x_3)x_2 + (f-e_2)x_3, \\ u_2 = -qe_1 + e_1e_3 + (e_3 - q - hx_1)x_1 + \\ (1-bx_2)x_2 + (e_1+x_1)x_3 + \tau, \\ u_3 = (r-1)e_3 - e_1e_2 - (e_2+g+x_2)x_1 - \\ (e_1+d)x_2 + (r-c)x_3. \end{cases} \quad (20)$$

***Proof*:** Based on the Lyapunov second method, we construct a Lyapunov function:

$$V(e_1, e_2, e_3) = \tfrac{1}{2}\sum k_i e_i^2. \quad (21)$$

By calculating the derivative of $V(t)$ along the trajectories of the error system,

$$\dot{V}(t) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3. \quad (22)$$

Using Equation (20) in Equation (13) and then in Equation (22) gives

$$\dot{V}(t) = e_1(-e_1) + e_2(-e_2) + e_3(-e_3),$$
$$\dot{V}(t) = -e_1^2 - e_2^2 - e_3^2,$$
$$\dot{V}(t) < 0. \quad (23)$$

As the time,t, tends to ∞, the error function tends to zero; that is, the complete synchronization between Equation (1) and Equation (11) is achieved. This completes the proof.

Anti-Synchronization

The anti-synchronization error is defined by

$$e_i = y_i + x_i, (i = 1,2,3). \quad (24)$$

Then, the anti-synchronization error of Equation (1) and Equation (11) is obtained as

$$\begin{cases} \dot{e}_1 = p(e_2 - e_1) + e_2e_3 - (a-p-x_2)x_1 \\ -(p+e_3-x_3)x_2 + (f-e_2)x_3 + u_1, \\ \dot{e}_2 = qe_1 - e_2 - e_1e_3 - (q - e_3 + hx_1)x_1 + \\ (1-bx_2)x_2 + (e_1 - x_1)x_3 + \tau + u_2, \\ \dot{e}_3 = e_1e_2 - re_3 - (e_2 + g - x_2)x_1 - \\ (e_1+d)x_2 - (c-r)x_3 + u_3. \end{cases} \quad (25)$$

The nonlinear active control law for u_i, $(i = 1,2,3)$ should be constructed in such a manner that the error dynamics Equation (25) is globally stable.

Choosing the control function $u = [u_1, u_2, u_3]^T$ as

$$\begin{cases} u_1 = -e_2e_3 + (a-p-x_2)x_1 + (p+ \\ e_3 - x_3)x_2 - (f-e_2)x_3 + v_1, \\ u_2 = e_1e_3 + (q - e_3 + hx_1)x_1 - (1-bx_2)x_2 \\ -(e_1 - x_1)x_3 - \tau + v_2, \\ u_3 = -e_1e_2 + (e_2 + g - x_2)x_1 + (e_1+d)x_2 \\ +(c-r)x_3 + v_3. \end{cases} \quad (26)$$

where v_1, v_2 and v_3 are the linear control input chosen such that Equation (25) becomes stable.

Follow the same procedure as in subsection 3.1, then, Equation (1) and Equation (11) will achieve anti-synchronization when the nonlinear active control is chosen as

$$\begin{cases} u_1 = (p-1)e_1 - pe_2 - e_2e_3 + (a-p- \\ x_2)x_1 + (p+e_3-x_3)x_2 - (f-e_2)x_3, \\ u_2 = -qe_1 + e_1e_3 + (q - e_3 + hx_1)x_1 - \\ (1-bx_2)x_2 - (e_1 - x_1)x_3 - \tau, \\ u_3 = (r-1)e_3 - e_1e_2 + (e_2+g-x_2)x_1 + \\ (e_1+d)x_2 + (c-r)x_3. \end{cases} \quad (27)$$

***Proof*:** Based on the Lyapunov second method, we construct a Lyapunov function:

$$V(e_1, e_2, e_3) = \tfrac{1}{2}\sum k_i e_i^2. \quad (28)$$

By calculating the derivative of $V(t)$ along the trajectories of the error system,

$$\dot{V}(t) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3, \quad (29)$$

and substituting Equation (27) in Equation (25) and then in Equation (29), one readily obtains

$$\dot{V}(t) = e_1(-e_1) + e_2(-e_2) + e_3(-e_3),$$
$$\dot{V}(t) = -e_1^2 - e_2^2 - e_3^2,$$
$$\dot{V}(t) < 0. \quad (30)$$

As the time,t, tends to ∞, the error function tends to zero; that is, the anti-synchronization between Equation (1) and Equation (11) is achieved. This completes the proof.

Hybrid synchronization

The hybrid synchronization error is defined by

$$e_1 = y_1 - x_1, e_2 = y_2 + x_2, e_3 = y_3 - x_3. \quad (31)$$

Then, the hybrid synchronization error of Equation (1) and Equation (11) is obtained as

$$\begin{cases} \dot{e}_1 = p(e_2 - e_1) + e_2e_3 + (a-p-x_2)x_1 - \\ (p+e_3+x_3)x_2 - (f-e_2)x_3 + u_1, \\ \dot{e}_2 = qe_1 - e_2 - e_1e_3 - (e_3 - q + hx_1)x_1 + \\ (1-bx_2)x_2 - (e_1 + x_1)x_3 + \tau + u_2, \\ \dot{e}_3 = e_1e_2 - re_3 + (e_2 + g - x_2)x_1 - \\ (e_1-d)x_2 - (r-c)x_3 + u_3. \end{cases} \quad (32)$$

The nonlinear active control law for u_i, $(i = 1,2,3)$ should be constructed in such a manner that the error dynamics Equation (32) is globally stable.

Choosing the control function $u = [u_1, u_2, u_3]^T$ as

$$\begin{cases} u_1 = -e_2e_3 - (a - p - x_2)x_1 + (p + e_3 + x_3)x_2 + (f - e_2)x_3 + v_1, \\ u_2 = e_1e_3 + (e_3 - q + hx_1)x_1 - (1 - bx_2)x_2 + (e_1 + x_1)x_3 - \tau + v_2, \\ u_3 = -e_1e_2 - (e_2 + g - x_2)x_1 + (e_1 - d)x_2 + (r - c)x_3 + v_3, \end{cases} \quad (33)$$

where v_1, v_2 and v_3 are the linear control input chosen such that Equation (32) becomes stable.

Follow the same procedure as in subsection 3.1, then, Equation (1) and Equation (11) will achieve hybrid synchronization when the nonlinear active control is chosen as

$$\begin{cases} u_1 = (p - 1)e_1 - pe_2 - e_2e_3 - (a - p - x_2)x_1 + (p + e_3 + x_3)x_2 + (f - e_2)x_3, \\ u_2 = -qe_1 + e_1e_3 + (e_3 - q + hx_1)x_1 - (1 - bx_2)x_2 + (e_1 + x_1)x_3 - \tau, \\ u_3 = (r - 1)e_3 - e_1e_2 - (e_2 + g - x_2)x_1 + (e_1 - d)x_2 + (r - c)x_3. \end{cases} \quad (34)$$

***Proof*:** Based on the Lyapunov second method, we construct a Lyapunov function:

$$V(e_1, e_2, e_3) = \frac{1}{2}\sum k_i e_i^2. \quad (35)$$

By calculating the derivative of $V(t)$ along the trajectories of the error system,

$$\dot{V}(t) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3, \quad (36)$$

and substituting Equation (34) in Equation (32), and then in Equation (36) one readily obtains

$$\dot{V}(t) = e_1(-e_1) + e_2(-e_2) + e_3(-e_3),$$
$$\dot{V}(t) = -e_1^2 - e_2^2 - e_3^2,$$
$$\dot{V}(t) < 0. \quad (37)$$

As the time, t, tends to ∞, the error function tends to zero; that is, the hybrid synchronization between Equation (1) and Equation (11) is achieved. This completes the proof.

RESULTS AND DISCUSSION

Numerical simulations are performed by the ODE45 algorithm embedded in MATLAB software package to establish the designed controller's effectiveness and feasibility. The initial conditions $(x_1, x_2, x_3) = (1, -3, 5)$ for the master system; and $(y_1, y_2, y_3) = (-5, 2, 1)$ for the slave system.

Complete Synchronization of Non-Identical Financial Systems

The complete synchronization of Equation (1) and Equation (11) is achieved using the controller in Equation (20).

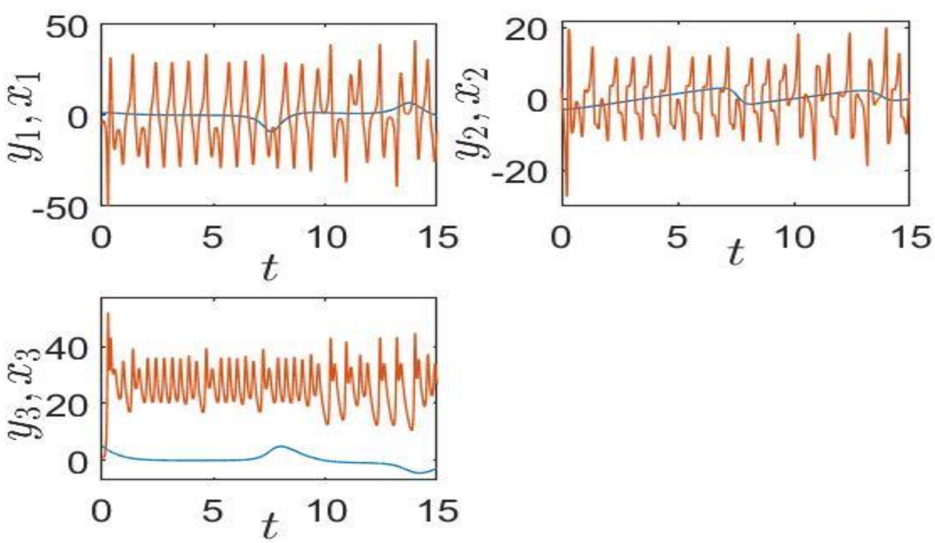

Figure 3: Time series of Equation (1) and Equation (11) in the absence of the controller

Figure 3 illustrates how state variables move chaotically with time when the controller is deactivated, demonstrating that these systems depend sensitively on initial conditions in the absence of the controller - a significant characteristic of a chaotic system.

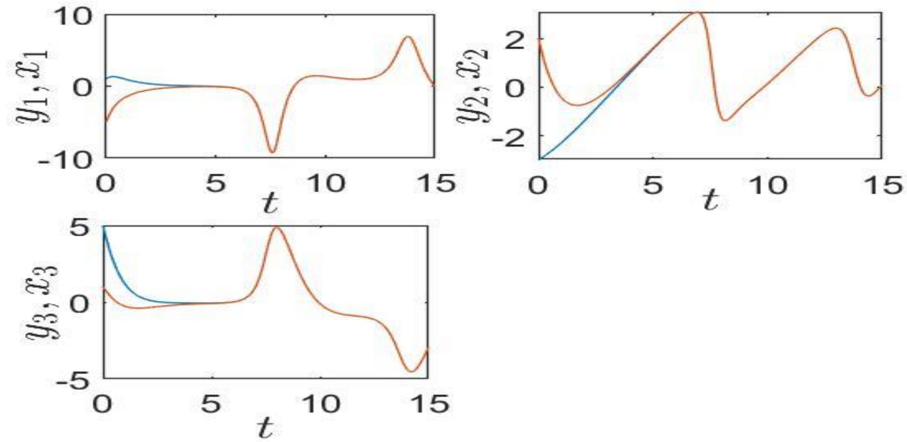

Figure 4: Time series of Equation (1) and Equation (11) when the controller is activated

Figure 4 illustrates how the system becomes asymptotically stable when controllers are activated.

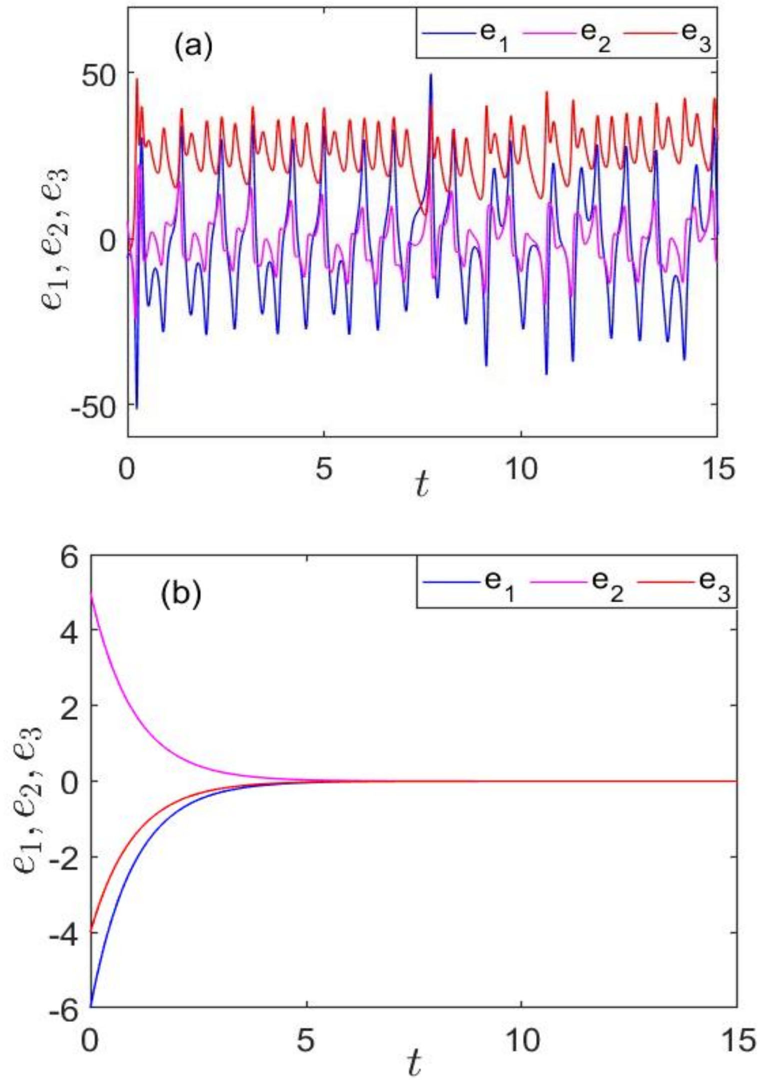

Figure 5: (a) Error dynamics between Equation (1) and Equation (11) in the absence of the controller. (b) Error dynamics between Equation (1) and Equation (11) when the controller is activated.

In Figure 5a, it can be seen that the variable errors do not synchronize with time in the absence of the controller but do so at $t \geq 5.5$when the controller is activated, as depicted in Figure 5b. This is confirmed by the synchronization quality, $e_{average} = \sqrt{e_1^2 + e_2^2 + e_3^2}$shown in Figure 6.

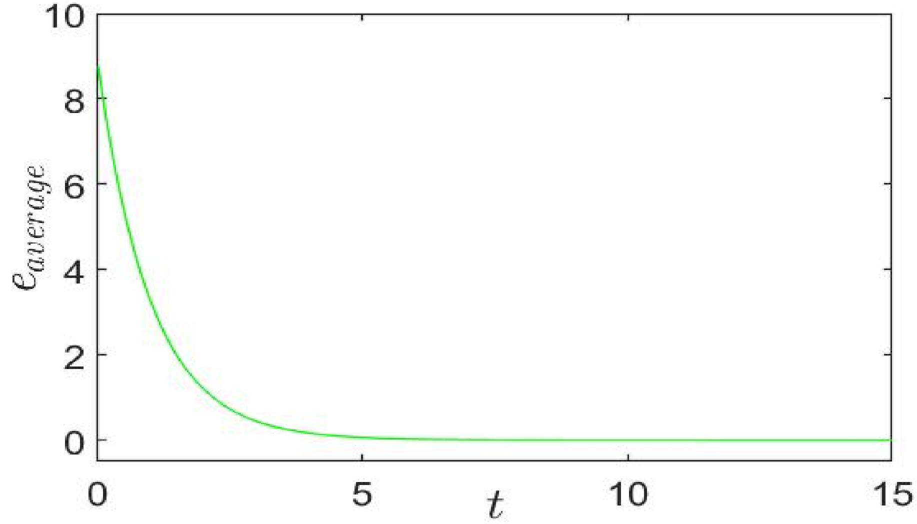

Figure 6: Synchronization quality between Equation (1) and Equation (11)

Anti-Synchronization of Non-Identical Financial Systems

The controller in Equation (27) is used to achieve the anti-synchronization of Equation (1) and Equation (11). When the controller is deactivated, state variables evolve chaotically over time as seen in Figure 7.

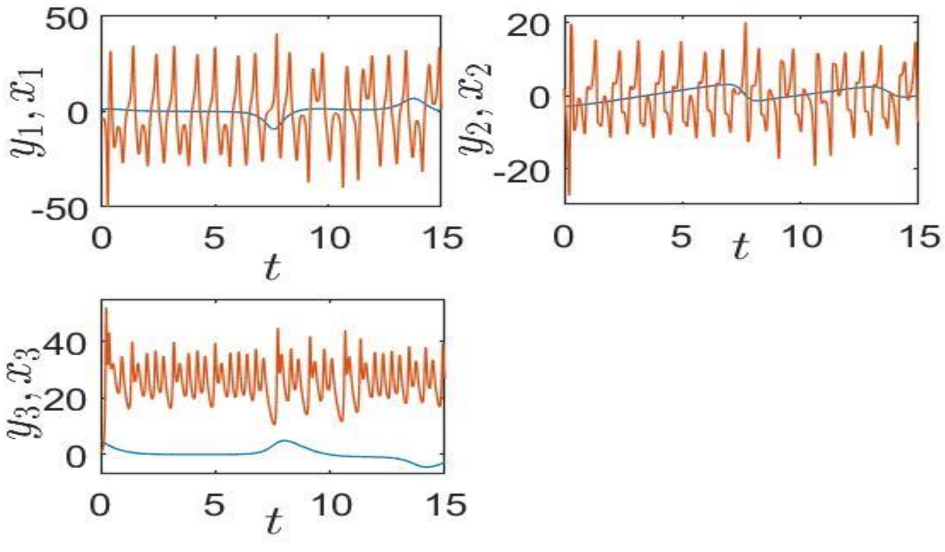

Figure 7: Time series of Equation (1) and Equation (11) in the absence of the controller

When controllers are activated, the system asymptotically stabilizes, as depicted in Figure 8.

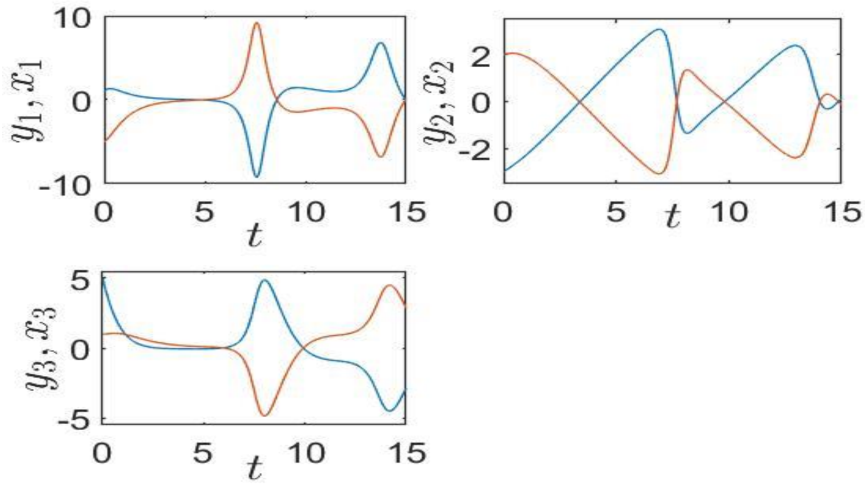

Figure 8: Time series of Equation (1) and Equation (11) when the controller is activated

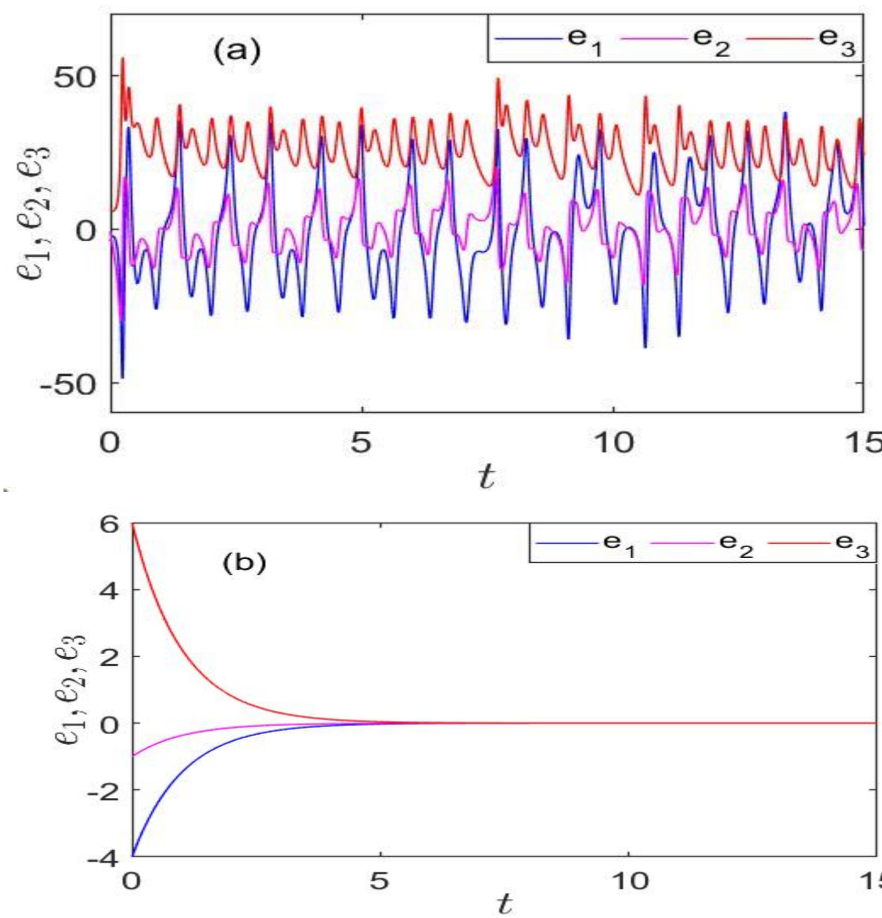

Figure 9: (a) Error dynamics between Equation (1) and Equation (11) in the absence of the controller. (b) Error dynamics between Equation (1) and Equation (11) when the controller is activated

Also from Figure 9a, it can be seen that the variable errors do not synchronize with time in the absence of the controller but do so at $t \geq 5.5$when the controller is activated, as depicted in Figure 9b. The synchronization quality $e_{average} = \sqrt{e_1^2 + e_2^2 + e_3^2}$, displayed in Figure 10, validates this.

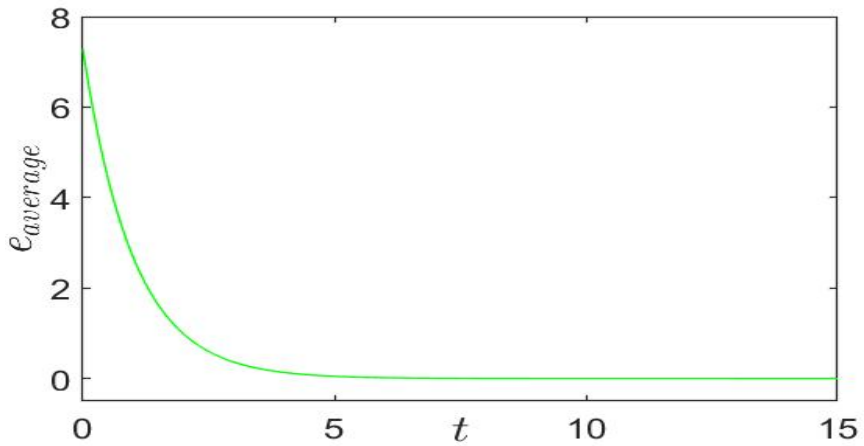

Figure 10: Synchronization quality between Equation (1) and Equation (11)

Hybrid Synchronization of Non-Identical Financial Systems

The hybrid synchronization of Equation (1) and Equation (11) is achieved by the controller in Equation (34).

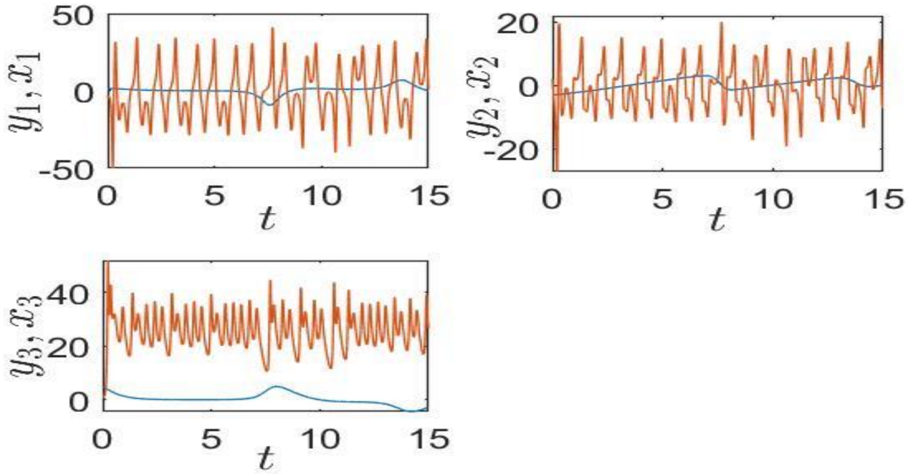

Figure 11: Time series of Equation (1) and Equation (11) in the absence of the controller

As demonstrated in Figure 11, state variables change chaotically over time when the controller is deactivated. In Figure 12, the system asymptotically stabilizes when controls are activated.

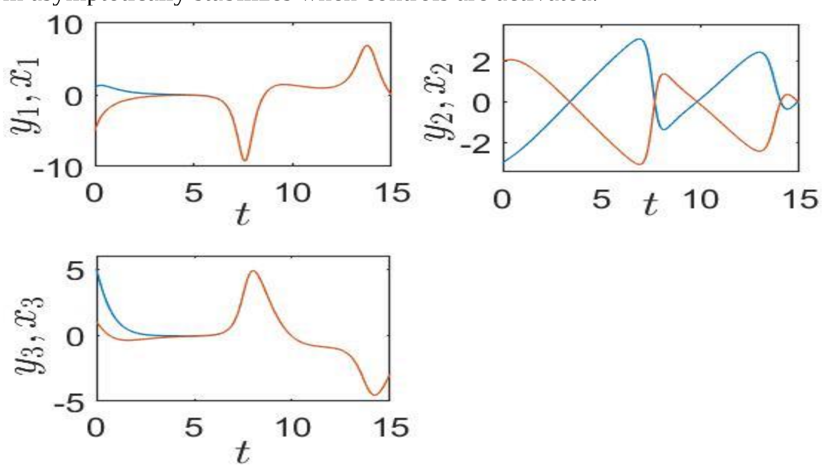

Figure 12: Time series of Equation (1) and Equation (11) when the controller is activated

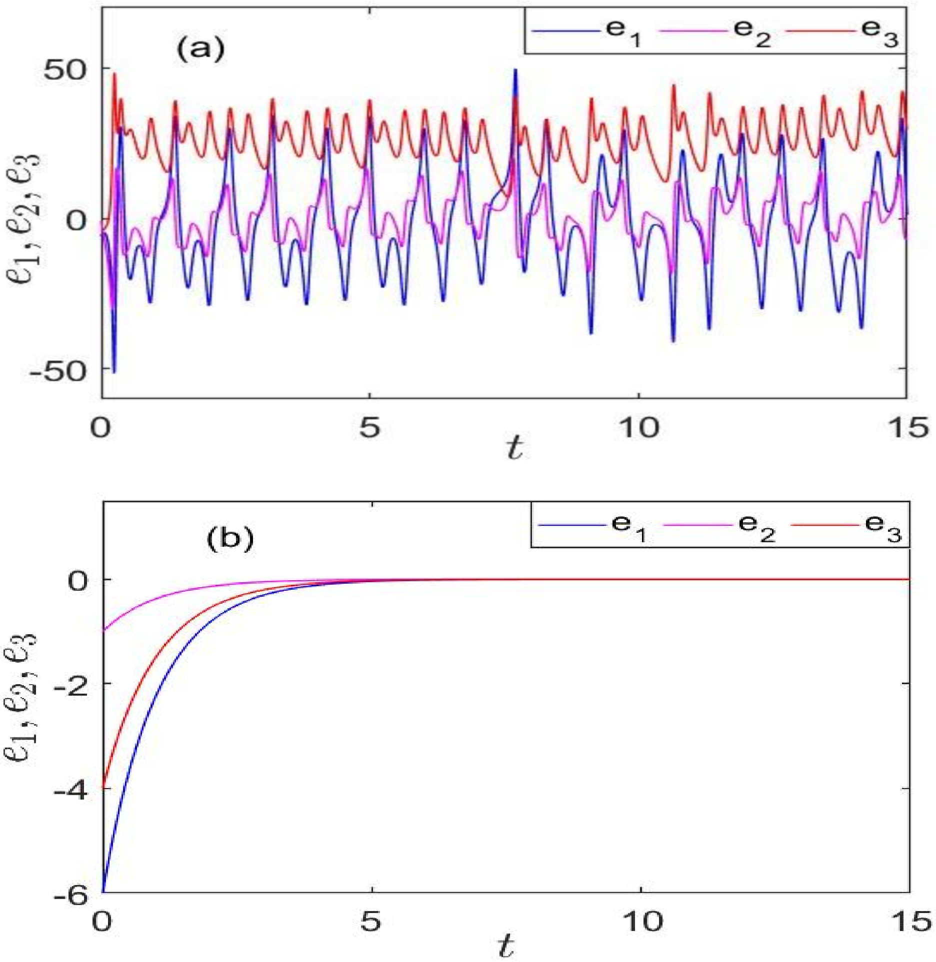

Figure 13: (a) Error dynamics between Equation (1) and Equation (11) in the absence of the controller. (b) Error dynamics between Equation (1) and Equation (11) when the controller is activated

Also from Figure 13a, it can be seen that the variable errors do not synchronize with time in the absence of controllers but do so at $t \geq 4.8$when the controller is activated, as depicted in Figure 13b. This is verified by the synchronization quality $e_{average} = \sqrt{e_1^2 + e_2^2 + e_3^2}$, which is shown in Figure 14.

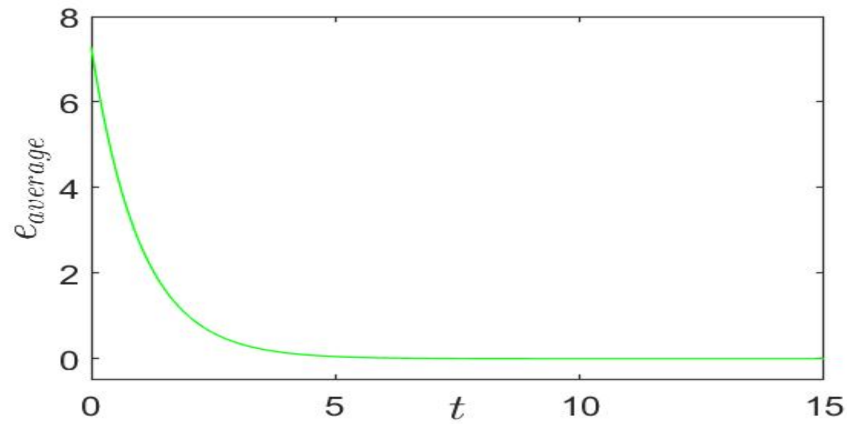

Figure 14: Synchronization quality between Equation (1) and Equation (11)

CONCLUSION

In this paper, the chaos synchronization of two non-identical finance systems is examined. The complete synchronization, the anti-synchronization and the hybrid synchronization are achieved via active control techniques. The linear system theory and Lyapunov second method are the two analytical approaches used to achieve the stability of error dynamics in each case. Also, relevant variables from the response and the drive systems are used to construct the controllers. The viability of the developed controllers is demonstrated through numerical simulations. Consequently, chaotic financial systems can be synchronized and stabilized by using a combination of various financial market indicators as a control.

REFERENCES

Aguilar-LOpez, R., Martnez-Guerra, R., Perez-Pinacho, C. (2014). Nonlinear observer for synchronization of chaotic systems with application to secure data transmission. *European Physical Journal, Special Topics*, **223**, 1541. https://doi.10.1140/epjst/e2014-02116-0.

Arashi, M., Rounaghi, M. (2022). Analysis of market efficiency and fractal feature of NASDAQ stock exchange: Time series modeling and forecasting of stock index using ARMA-GARCH model. *Future Business Journal*, **8**, 1. https://doi.10.1186/s43093-022-00125-9.

Baleanu, D., Jajarmi, A., Sajjadi, S. S., Asad, J. H. (2020). The fractional features of a harmonic oscillator with position-dependent mass. *Commun. Theor.*, **72**, 1. https://doi.10.1088/1572-9494/ab7700.

Bourdeau-Brien, M., Kryzanowski, L. (2017). The impact of natural disasters on the stock returns and volatilities of local firms. *The Quarterly Review of Economics and Finance*, **63**, 259. https://doi.10.1016/j.qref.2016.05.003.

Bowang, S., Kakmeni, F. M. M., Siewev, M. S. (2007). Secure communication via parameter modulation in a class of chaotic system. *Journal of Commun. Nonlinear Sci. Numer. Simulation* **12**(03), 397. https://doi.10.1016/j.cnsns.2005.03.002.

Dousseh, Y. P., Monwanou, A. V., Koukpemedji, A. A., Miwadinou, C., Chabi Orou, J. B. (2022). Dynamics analysis, adaptive control, synchronization and anti-synchronization of a novel modified chaotic financial system. *International Journal of Dynamics and Control*, **11**, 1. https://doi.10.1007/s40435-022-01003-6.

Eisencraft, M., Fanganiello, R. D.,Grzybowski, J. M. V., Soriano, D. C., Attux, R., Batista, A. M., Macau, E. E. N., Monteiro, L. H. A., Romano, J. M. T., Suyama, R., Yoneyama, T. (2012). Chaos-based communication systems in non-ideal channels. *American Jornal of Physics*, **17**(12), 4707. https://doi.10.1016/j.cnsns.2011.05.030.

Filali, M. B., Pierre, B. (2014). On observer-based secure communication design using discrete-time hyperchaotic systems. *Communications in Nonlinear Science and Numerical Simulation*, **19**(05), 1424. https://doi.10.1016/j.cnsns.2013.09.005.

Freeman, W. J. (1992). Tutorial on neurobiology: from single neurons to brain chaos. *International Journal of Bifurcation and Chaos*, **2**(03), 451. https://doi.10.1142/S0218127492000653.

Fujisaka, H., Yamada, T. (1983). Stability theory of synchronized motion in coupled oscillator systems. *Progress of Theoretical Physics*, **69**(01), 32. https://doi.10.1143/PTP.69.32.

Gabriel, P., Hilda, A. C. (1995). Extracting messages masked by chaos. *Phys. Rev. Lett.*, **74**, 1970. https://doi.10.1103/PhysRevLett.74.1970.

Gan, X. Wang, H., Yuan, R., Ao, P. (2021). A new criterion beyond divergence for judging the dissipation of a system: dissipative power. *Frontiers in Physics*, **9**, 458. https://doi.10.3389/fphy.2021.695489.

Hsieh, D. A. (1991). Chaos and nonlinear dynamics: Application to financial markets. *The Journal of Finance*, **46**, 1839. https://doi.10.2307/2328575.

Juarez, F. (2015). Chaos and Complexity in Financial Statements. *Information Resources Management Association, IGI Global*, 1399--1430. https://doi.10.4018/978-1-4666-8468-3.

Kilikevicius, A., Jurevicius, M., Bureika, G., Turia, V. (2015). Effect of external excitation on dynamic characteristics of vibration isolating table. *Maintenance and Reliability*, **17**(2), 260. https://doi.10.17531/ein.2015.2.13.

Klioutchnikov, I., Sigova, M., Beizerov, N. (2017). Chaos Theory in Finance. *Procedia Computer Science*, 119, 368. https://doi.10.1016/j.procs.2017.11.196.

Liao, Y., Zhou, Y., Xu, F., Shu, X. (2020). A Study on the Complexity of a New Chaotic Financial System. *Complexity*, **1**, 8821156. https://doi.10.1155/2020/8821156.

Lu, X. (2020). A financial chaotic system control method based on intermittent controller. *Mathematical Problems in Engineering*, 5810707. https://doi.10.1155/2020/5810707.

Metescu, A. (2022). Fractal market hypothesis vs. efficient market hypothesis: Applying the R/S analysis on the Romanian market. *Journal of Public Administration, Finance and Law*, **11**, 199. https://doi.10.47743/jopafl-2022-23-17.

Olusola, O. I., Oyeleke, K. S., Vincent, U. E., Njah, A. N. (2020). Secure Communication Scheme Based on Synchronization of Non-Identical Hyperchaotic Systems. *Journal of Applied Nonlinear Dynamics*, **9** (2), 273. https://doi.10.5890/JAND.2020.06.009.

Pecora, L. M., Carroll, T. L. (1990). Synchronization in chaotic Systems. *Phys. Rev. Lett.* **64**, 821. https://doi.10.1063/1.4917383.

Pikovsky, A. S., Rosemblum, M., Kurths, J. (2002). Synchronization: A universal concept in nonlinear science. *American Jornal of Physics* **70**(6). https://doi.10.1119/1.1475332.

Rameika, R. (2007). New results from accelerator neutrino experiments. *International Journal of Modern Physics A*, **22**(30), 5544. https://doi.10.1142/S0217751X07038803.

Ren, H. P., Baptista, M. S., Grebogi, C. (2013). Wireless communication with chaos. *Phys. Rev. Lett.*, **110**, 18410. https://doi.10.1103/PhysRevLett.110.184101.

Tusset, A., Fuziki, M., Balthazar, J., Andrade, D., Lenzi, G. G. (2023). Dynamic analysis and control of a financial system with chaotic behavior including fractional order. *Fractal and Fractional*, **7**, 535. https://doi.10.3390/fractalfract7070535.

Wen, S., Shen, Y., Yang S., Wang, J. (2017). Dynamical response of Mathieu-Duffing oscillator with fractional-order delayed feedback. *Chaos, Solitons and Fractals*, **94**, 54. https://doi.10.1016/j.chaos.2016.11.008.

Wu, Z., Xie, J., Fang, Y., Xu, Z. (2007). Controlling chaos with periodic parametric perturbations in Lorenz system. *Chaos, Solitons and Fractals*, **32**, 104. https://doi.10.1016/j.chaos.2005.10.060.

Woolf, P. J. (2009). Chemical Process Dynamics and Controls, Open textbook library, University of Michigan Engineering Controls Group, Ann Arbor, United States, 447--458. https://eng.libretexts.org.

Xiao-Dan, Z., Liu, X.,Yuan, Z., Cheng, L. (2013). Chaotic dynamic behaviour analysis and control for a financial risk system. *Chinese Physics B*, **22**, 030509. https://doi.10.1088/1674-1056/22/3/030509.