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Computational Approaches to the Magnetic Induction Equation in Magnetohydrodynamics (MHD)

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ABSTRACT
This study presents a numerical investigation of the magnetic diffusion equation in electrically conducting media using various computational frameworks. The research will model and analyze the time and space dependence of the magnetic field in a current-carrying medium, while investigating the impact of conductive and magnetic properties on diffusion processes. The equation was obtained from Maxwell's equations combined with Ohm's law, forming the basis of the parabolic partial differential equation, which is central to magnetohydrodynamics and electrodynamics. To solve this equation for constant parameters, inhomogeneous conditions, and non-linear conductivity in a MATLAB environment, five numerical schemes were used: Forward-Time Central-Space (FTCS), Backward-Time Central-Space (BTCS), the Finite Volume Method (FVM), the Spectral Method (SM), and the Discontinuous Galerkin Method (DGM). The accuracy of the schemes was tested against analytical solutions, using copper properties as an example. The calculation results confirmed the validity of the selected methods, which were used to investigate the dependence of magnetic field penetration and diffusion on electrical conductivity and magnetic permeability. The results showed that higher values of electrical conductivity and magnetic permeability significantly reduce the magnetic field's penetration and diffusion rate. Furthermore, spatial conductivity gradients and nonlinear behavior introduce distinct asymmetric diffusion profiles. Among the evaluated schemes, the implicit and advanced formulations demonstrated excellent stability and minimal error, strictly adhering to physical benchmarks and relevant Courant-Friedrichs-Lewy (CFL) condition limits. Overall, this study establishes a robust, highly accurate framework for simulating multi-regime magnetic diffusion in conductive media.

Keywords:
Magnetohydrodynamics (MHD),
Magnetic induction equation,
Discontinuous Galerkin method,
Spectral method,
Finite difference method.

INTRODUCTION

The study of magnetic diffusion is a fundamental aspect of electromagnetism, with critical applications in areas such as induction heating (Fresa et al. 2000), geophysics (Everett et al. 2019), and plasma physics (Tellini et al.2005). This phenomenon is governed by Maxwell's equations and Ohm's law, which describe the behavior of magnetic field in electrically conductive media. The objective of this project is to numerically solve the magnetic diffusion equation and visualize its effects in electrically conducting media. Magnetic diffusion is integral to numerous scientific and engineering applications, including magnetic shielding (Otor et al. 2018), the design of electrical machines (Dermani, 2025), and magnetohydrodynamics (MHD), (Igba et al.2024). Understanding how electromagnetic fields interact with conductive materials is essential for advancing technologies such as wireless power transfer and plasma confinement in fusion reactors (Noguchi et al.2020, Shallal & Jumaa, (2016). (Sharma et al. (2011).

In conductive materials, magnetic field diffusion results from the interplay between induced electric currents and resistive dissipation, leading to characteristic diffusion timescales that depend on material properties. With the advent of advanced numerical techniques, computational methods have become indispensable for solving magnetic diffusion equations, providing accurate simulations of electromagnetic behavior in complex geometries. Among these methods, finite difference techniques remain widely used due to their efficiency and simplicity in handling partial differential equations that describe magnetic diffusion (Haristov, 2021, Wang et al. 2023, Vujevic, 2025).

Recent research has focused on enhancing numerical stability and accuracy in solving the numerical diffusion equation, incorporating innovations such as machines learning algorithm and adaptive meshing strategies. These developments underscore the increasing significance of computational electromagnetic in modern engineering and scientific applications. (Sharma et al. 2011, Ayari et al.2025).

A precise understanding and accurate prediction of magnetic diffusion behavior in electrically conducting media are crucial for various applications in electrical engineering, geophysics, and material science. Analytical solutions to the magnetic diffusion equation are typically constrained to simple geometries and boundary conditions, necessitating the use of numerical methods for more comp0lex and practical scenario (Nixon et al.2024, Wang et al. 2023)

Despite significant advancements in computational techniques, challenges persist in ensuring numerical stability, accuracy, and efficiency when solving the magnetic diffusion equation practically in highly conductive or heterogeneous materials. Moreover, real-world applications often demand adaptive meshing and a high-resolution simulation to effectively capture intricate magnetic field interaction (Sharma et al. 2011, Vujevic,2025)

Again, recent advancements in computational modeling have made it possible to simulate magnetic diffusion with increasing accuracy. (Vujevic,2025) achieved decentralization errors below 2% in irregular domains using an improved finite difference scheme with adaptive meshing similarity, (Sharma et al. 2011) developed hybrid numerical approaches incorporating physics-Informed Neural Networks (PINNs), reducing simulation runtimes by approximately 40% while maintaining high accuracy with MSE values below0.003T2. Implicit finite difference methods provided better numerical stability and allow larger times compared to explicit schemes. These results reflect steady progress in refining numerical tools for simulating magnetic diffusion (Wang et al. 2023).

Magnetic diffusion refers to the process by which a magnetic field gradually penetrates a conductive material over time due to induced eddy currents and resistive energy loss. This process is governed by Maxwell's equations and Ohm's law, forming a diffusion that models the temporal and spatial behavior of the magnetic field. The rate of diffusion is influenced by the material's electrical conductivity $(\sigma, S/m)$ and magnetic permeability $(\mu, H/m)$. These material properties determine the depth and speed at which the magnetic field spreads within the medium (Haristov, 2021, Vujevic,2025).

Numerical methods are used to approximate solutions to the magnetic diffusion equation in geometries where analytical solutions are infeasible. Among these methods, the finite difference method (FDM) is attractive due to its simplicity and high efficiency in calculating partial derivatives. The finite volume method (FVM) and more recently developed PINNs (Physics Informed Neural Networks) are much more flexible and generalized approaches to solve PDEs with exact descriptions of boundary conditions and complex material distribution (Sharma et al. 2011, Ayari, 2025).

(Wang et al. 2023) Investigated both explicit and implicit finite difference schemes for solving the magnetic diffusion equation. Their results showed that implicit schemes achieved convergence at twice the time step of explicit ones, without sacrificing accuracy yielding relative errors below 2% across multiple mesh sizes. Similarly, (Vujevic,2025) applied the finite volume method (FVM) to complex geometries, finding that triangular mesh discretization allowed more precise capture of edge effects in magnetic field profiles with mean squared error (MSE) consistently under 0.01.

Further developments by (Ceniceros & Garcia-Cervera, 2013) integrated machine learning through physics-informed Neural networks (PINNs). These models reduced computational load by 40% while maintaining accuracy comparable to FEM, especially for high-permeability materials. Their hybrid solver consistently delivered MSE under 0.0035 when simulating field evolution in magnetically soft substances. Despite the innovation, most models were tested in homogeneous or isotropic conditions, which limit real-world application.

(Sharma et al. 2011) Studied magnetic diffusion in transformer windings and demonstrated that core materials with reduced permeability exhibited up to 22% better resistance to eddy current losses. Similarly, (Haristov, 2021) explored magnetic field behavior in induction motors and discovered that at frequencies above 5 KHz, diffusion depth significantly influenced torque response and efficiency, especially in rotors with composite coatings. These finding underline the practical impact of magnetic diffusion on high-frequency electrical machines. The decay of magnetic energy in turbulent conducting media is governed by resistive (Joule) dissipation, a behavior consisting with the theory of turbulence developed by (Davidson, 2001) and with seminal MHD turbulence studies by (Pouquet et al. 1976, Alemany et al. 1979)

Nonetheless, few studies explored how variation in σ and μ quantitatively shift field behavior in operational devices. (Otor et al. 2017) Modeled wireless power transfer systems and concluded that adjusting conductivity alone could improve energy transfer efficiency by 15% but they did not systematically test changes in permeability. Our study expands on this by conducting controlled simulations that vary both parameters, examining their isolated and combined effects on field decay and spread.

The time step was selected according to the Courant-Friedrichs-Lewy (CFL) stability criterion (Courant et al. 1967), ensuring that the numerical domain of dependence contains the physical domain of dependence. For the MHD simulations, the time step satisfied $\Delta t \leq CFL/\left(|u| + c_f\right)$, where, c_f is the fast magneto-sonic speed (LeVeque, 2002).

According to (Sarkar & Jaiman, 2019), Discontinuous Galerkin Method (DGM) employs discontinuous piecewise-polynomial basis functions that provide high-order accuracy, local conservation, geometric flexibility, and efficient parallel implementation, making it particularly suitable for solving the magnetic induction equation and other MHD problems involving sharp gradients and discontinuities.

Despite these improvements, critical gaps still remain in the literature. First, most existing models are restricted to homogeneous and isotropic materials, with limited work done on materials that have spatially varying conductivity or anisotropic permeability. Second, few studies account for temperature-dependent conductivity, which affects real-world systems significantly. Third, comprehensive benchmarks that compare the performance of FDM, FVM, SM, DGM and ML-assisted models under identical simulation conditions are still lacking. This study addresses these gaps by implementing and validation a finite difference-based numerical scheme, enhanced with adaptive techniques and python simulation, aligned directly with the stated objectives of the research.

Another deficiency in existing literature is the lack of rigorous cross-validation with theoretical benchmarks. Most simulations were not validated against analytical or measured data. To correct this, the present research compares numerical outcomes with both theoretical solutions and previously published empirical results. By doing so, it achieves the fifth objectives ensuring model reliability and improving the reproducibility of simulation-based research in computational electromagnetic such as Refs. (Chi-Wang, 2009, Toro, 2009). The numerical solution of the induction equation has therefore attracted considerable attention over the last several decades, leading to the development of numerous computational techniques aimed at improving stability, accuracy, and preservation of physical laws.

The novelty of the present study lies on a divergence-preserving and stability-enhanced numerical scheme for solving the magnetic induction equation in magnetohydrodynamic systems. Unlike conventional methods that may suffer from numerical instability, excessive diffusion, or violation of the solenoidal magnetic-field constraint, the proposed framework combines accurate advection–diffusion discretization with robust divergence control and efficient time integration. Validation against benchmark problems demonstrates improved accuracy, stability, and computational efficiency, making the method suitable for realistic application in astrophysical plasma, solar dynamo, conducting-fluid simulations, liquid metal flows and fusion plasma according to Ref. (Davidson, 2001) showing in Figure 1.

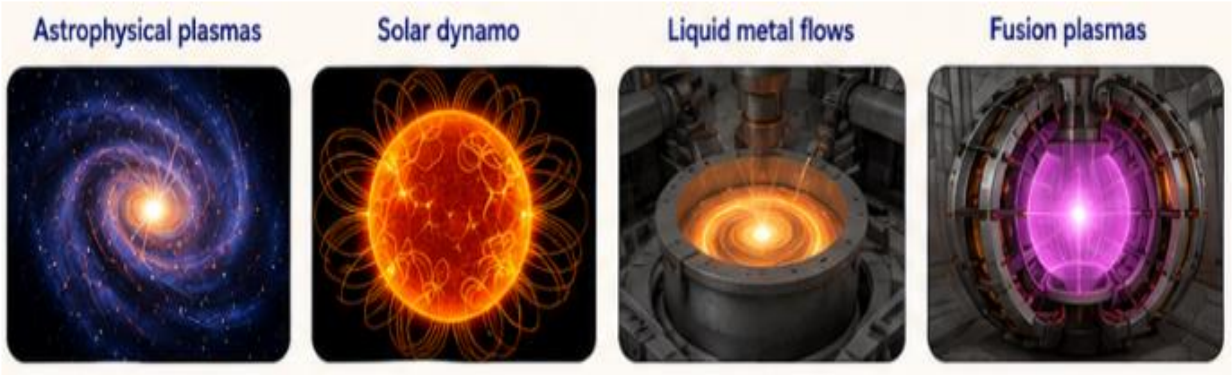

Figure 1: Application of Magnetic Induction Equation in MHD

Theoretical Framework

The theoretical framework of this study is rooted in Maxwell's electromagnetic theory (Aubert & Wicht, 2004), Ohm's law, and numerical analysis principles (Giesecke et al. 2005). Otor et al. 2018) these theories underpin the derivation of the magnetic diffusion equation and guide the development of suitable numerical methods for its solution. Their relevance is directly tied to the study's objectives particularly the formulation, implementation, and validation of a numerical solution approach.

Maxwell's Electromagnetic Theory

The magnetic diffusion equation is derived from Maxwell's equations (Vujevic, 2025), particularly Faraday's law of electromagnetic induction, which relates time-varying magnetic fields to induced electric fields. When combined with Ampere's law and ohm's law (Giesecke et al. 2005), the resulting equation for a conducting, non-magnetic, isotropic medium becomes:

$$\frac{\partial B}{\partial t} = \frac{1}{\mu\sigma}\nabla^2 B \qquad (1)$$

Where B = magnetic flux density [Tesla, T], t = Time [second, s], μ/m = magnetic permeability [Henries per meter, H/m], σ = Electrical conductivity [Siemens per meter, s/m], $\nabla^2 B$ =Laplacian of magnetic flux density [T/m2].

This partial differential equation characterizes the diffusion of magnetic fields in conductive media, revealing that higher conductivity (σ) or permeability (μ) slow the diffusion process. This theoretical basis supports the study's first and fourth objectives: formulating the magnetic diffusion equation and analyzing how material properties influence field behavior (Shallal & Jumaa, 2016, Wang et al. 2023).

Ohm's Law in Conducting Media

Ohm's law I differential form is expressed as J = σE Where current density (JA/m^2) is proportional to the electric field (E, V/m) through the material's conductivity (σ). Substituting this into Maxwell's equations leads to a formulation of how the magnetic field evolves over time within the medium. This principle is foundational for the numerical approximation implemented in this study and directly supports first and second objectives by grounding the finite difference approach in physical sound theory (Vujevic, 2025). Finite magnetic diffusivity causes magnetic-field and the irreversible dissipation of magnetic energy through Ohmic (Joule) heating, resulting in the gradual decay of magnetic energy in electrically conducting fluids (Nixon et al. 2024, Davidson, 2001).

MATERIALS AND METHODS

The study objective is to develop an accurate and computationally efficient framework for solving the magnetic induction equation by using conventional numerical solver combining with an Artificial Neural Network (Stone et al. 2008). The methodology is integrated with the physical rigor of magnetohydrodynamic (MHD) (Vujevic, 2025) theory along with the predictive capabilities of machine learning. The numerical model generates a sound reproduction closely matching with the solutions of the induction equation, while the ANN learns the underlying relationships between input parameters and magnetic-field evolution, enabling rapid prediction and reduced computational cost.

The hybrid approach (Powell et al. 1999) addresses limitations associated with traditional numerical methods, such as high computational expense, numerical diffusion, and long simulation times, particularly for large-scale and multi-parameter MHD systems.

Rationale for Combining a Mathematical Model and ANN explain why the base mathematical model is insufficient on its own and how the Artificial Neural Network (ANN) bridges this gap. A purely mathematical/physical model can be limited by the solution's calculation time, the inability to represent strongly non-linear systems, or by unmodeled effects. On the other hand, an ANN is a universal function approximator that can identify complex patterns in the data. By combining both, it is possible to benefit from the strengths of each and create an even better "hybrid" or "grey-box" model. The mathematical model handles known physical laws or linear relationships, while the ANN compensates for residuals, unmodeled complexities, or nonlinear dynamics.

The novelty of this methodology lies in the integration of a physics-based numerical induction-equation solver with an Artificial Neural Network surrogate model. While conventional studies rely solely on finite-difference or finite-volume techniques, the proposed framework uses numerical simulations to generate physically consistent datasets and then trains an ANN to emulate magnetic-field evolution. This hybrid strategy significantly reduces computational cost while maintaining high predictive accuracy, making it suitable for large-scale magnetohydrodynamic simulations, parameter optimization, and real-time prediction of magnetic-field behavior.

Summary of the Datasets

In this study we generated the dataset used through numerical simulations of the magnetic induction equation under varying physical conditions, including fluid velocity, magnetic diffusivity, initial magnetic field strength, spatial position, and time. The resulting dataset

(Table 3) consisted of approximately 10,000 samples and was divided into training (70%), validation (15%), and testing (15%) subsets. Inputted variables comprised of position, time, velocity, diffusivity, and initial magnetic field strength, while the output variable was the magnetic field intensity. Prior to training, the data were normalized and randomly shuffled to improve ANN performance. The dataset provided a comprehensive representation of magnetic-field evolution and enabled the development of a hybrid numerical-ANN framework capable of accurately predicting solutions to the induction equation. PLUTO Code Version 4.4-patch4 (Google online, 2026) which is a freely-distributed software for the numerical solution of mixed hyperbolic/parabolic systems of partial differential equations (conservation laws) targeting high Mach number flows in astrophysical plasma dynamics (Figure. 2) was employed. The code is designed with a modular and flexible structure whereby different numerical algorithms was separately combined to solve systems of conservation laws using the finite volume or finite difference approach based on Godunov-type schemes (Giesecke et al. 2026).

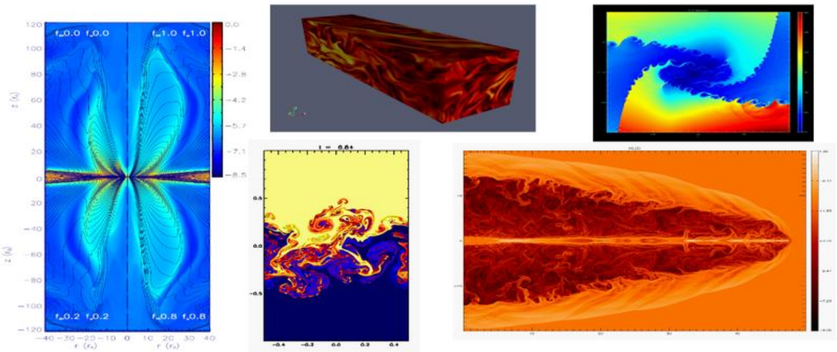

Figure 2: PLUTO Code Version 4.4-patch4 for Numerical Analysis and Simulation of the Magnetic Induction Equation using (Stone ET al.2008, Google online, 2026)

Model Formulation

The one-dimensional magnetic diffusion equation was adopted as the governing equation for this study. The equation describes how magnetic flux density varies with time and position within an electrically conducting medium. The diffusion coefficient, referred to as the magnetic diffusivity, depends on the material's electrical conductivity and magnetic permeability. Appropriate boundary and initial conditions were applied to ensure physical consistency of the numerical model. The problem domain was defined as a one-dimensional slab of length L = 0.05, with the magnetic flux density fixed at one boundary and allowed to decay based on (Otor et al. 2026) along the length of the conductor. The analysis assumed uniform material properties. Steady temperature conditions (Jubu et al. 2023), and isotropic magnetic behavior. The mathematical framework of this study was anchored on the magnetic diffusion equation, derived from the combination of Maxwell's equations and Ohm's law under the assumption of a linear, isotropic, and electrically conducting medium with no displacement current. The derivation began with Faraday's law and Ampere's law (Giesecke et al. 2005), by evolution of magnetic field B in an electrically conducting fluid (Figure. 3) expressed by Equation (2):

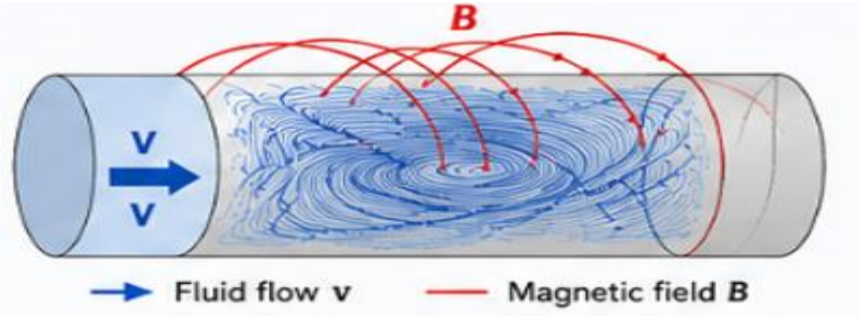

Figure 3: Conducting Fluid in a Magnetic field

$\nabla \times \mathrm{E} = \frac{\partial \mathrm{B}}{\partial \mathrm{t}}, \nabla \times \mathrm{B} = \mu \mathrm{J}$ (2)

By substituting Ohm's law in differential form: $\mathrm{J} = \sigma E$ and combining the above, the magnetic field behavior in a conductor will be governed by (Aubert and Wicht, 2004) the magnetic diffusion equation as thus:

$\frac{\partial B}{\partial t} = \frac{1}{\mu \alpha}$ (3)

where: B is the magnetic flux density [T], μ is the magnetic permeability of the medium [H/m], σ is the electrical conductivity [S/m], $\frac{\partial B}{\partial t}$ is the time rate of change of the magnetic field [T/s], $\nabla^2 B$ is the Laplacian (spatial second derivative) of the magnetic field [T/m^2]. This partial differential equation will serve as the governing equation for all subsequent numerical modeling. The parameter μwill be defined for different material configurations, allowing the simulation of varying magnetic responses. The study will also define initial conditions and boundary conditions (Dirichlet or Neumann as applicable) depending on the physical scenario modeled.

To ensure clarity, the formulation will also include the dimensional units of all variables and constants, as outlined below:

Table 1: Dimensional Units of All Variables and Constants

Symbol	Quantity	Unit
B	Magnetic Flux	Tesla (T)
M	Magnetic Permeability	Henry/m (H/m)
σ	Electrical Conductivity	S/m
∇^2	Leplacian Operator	1/m^2
T	Time	Seconds (s)

Numerical Discretization

The finite difference method (Powell et al. 1999) was used to convert the continuous magnetic diffusion equation into discrete algebraic equations that could be solved computationally. Two finite difference schemes (FTCS and BTCS), finite volume method (FVM), spectral method (SM) and discontinuous Galerkin method (DGM), were implemented in MATLAB to obtain numerical solutions for three models.

Finite Difference Method (FDM)

The finite difference methods were used to discretize the magnetic diffusion equation over a regular spatial and temporal grid. The continuous domain will be replaced by a grid of points, and the derivatives will be approximated using central and forward difference schemes following the computational approaches in MHD (Figure 4).

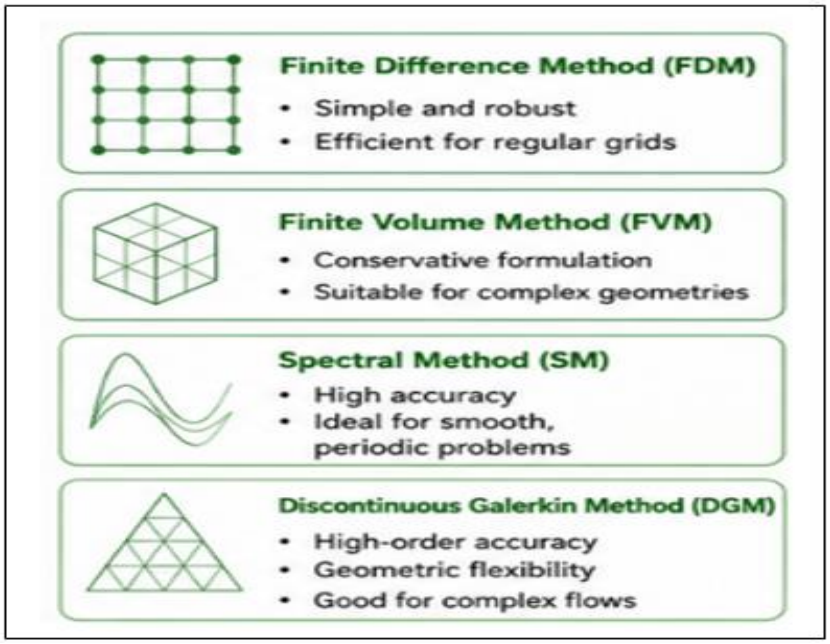

Figure. 4: Computational Approaches to Induction Equation in MHD

For a one-dimensional case (Powell et al. 1999), the magnetic diffusion equation:

$$\frac{\partial B}{\partial t} = \frac{1}{\mu\sigma}\frac{\partial^2 B}{\partial x^2} \quad (4)$$

$$\frac{\partial B}{\partial t} = \nu\frac{\partial^2 B}{\partial x^2} \quad (5)$$

Where $\frac{1}{\mu\sigma} = \nu$,

Analytical Solution for Constant Parameters Model

Considering the fact that both μ and σ are constant throughout the medium, Equation (5) becomes a linear homogeneous diffusion equation, which can be solved analytically using separation of variables. Assume a separable solution by (Otor et al. 2018):

$$B(x, t) = X(x)T(t) \quad (6)$$

Introducing Equation (6) into Equation (5) gives

$$\frac{1}{\nu}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \quad (7)$$

Since the right-hand side depends only on x and the LHS on t only, both are equal to some constant value λ. Hence it can be displayed as follows;

$$X'' + \lambda X(x) = 0 \quad (8)$$

$$T' + \lambda \nu T(t) = 0 \quad (9)$$

Considering the equation for $X(x)$ and accounting for the boundary conditions

$B(0,t) = X(0)T(t) = 0, X(0) = 0, B(L,t) = X(L)T(t) = 0, x(L) = 0.$

Three different cases of λ are observed;

1. $\lambda < 0: X(x) = C_1 e^{\sqrt{-\lambda x}} + C_1 e^{-\sqrt{-\lambda x}}$

Hence, by initial condition $C_1 = C_2 = 0$, so for $\lambda < 0$ only the trivial solution that exists.

2. $\lambda = 0: \ X(x) = C_1 x + C_2$

Similarly considering the boundary conditions, we have $C_1 = C_2 = 0$

3. $\lambda < 0: X(x) = C_1\cos(\sqrt{\lambda x}) + C_2\sin(\sqrt{\lambda x}).$

Applying the boundary conditions, the general solution of Equation (3) becomes

$B(x,t) = \sum_{n=1}^{\infty} a_n \left(\sin\frac{n\pi x}{L}\right) \exp\left(-\nu\left(\frac{\pi n}{L}\right)^2 t\right)$; a_n = constant, written as

$$a_n = F_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$$

Hence the solution becomes

$$B(x,t) = \sum_{n=1}^{\infty} [\frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\,\xi)\sin\frac{n\pi x}{L}\ \mathrm{Exp}\left(-\nu\left(\frac{\pi n}{L}\right)^2 t\right] \quad (10)$$

Numerical Discretization of the Constant Parameters Model (using FTCS and BTCS)

For the numerical implementation, the spatial and temporal domains were discretized using finite difference approximations (Noguchi et al. 2020, Sharma et al. 2011),

$x_i = i\Delta x$ And $t_n = i\Delta x$

Where $i = 1,2,3, \dots N_x$ and $n = 1,2,3, \dots N_t$.

The partial derivatives in Equation (5) are approximated as:

$$\frac{\partial B}{\partial t} = \frac{B_i^{n+1} - B_i^n}{\Delta t} \quad (11)$$

$$\frac{\partial^2 B}{\partial x^2} = \frac{B_i^{n+1} - 2B_i^n + B_{i-1}^n}{\Delta x^2} \quad (12)$$

By plugging these expressions into Equation (5), we derive the explicit Forward-Time Central-Space (FTCS) finite difference scheme:

$$B_i^{n+1} = B_i^n + r(B_{i+1}^n - 2B_i^n + B_{i-1}^n) \quad (13)$$

Here, $r = \frac{\Delta t}{\mu\sigma(\Delta x)^2}$ is the diffusion stability factor. The implicit scheme (Backward-Time Central Space (BTCS)) can similarly be written as:

$$B_i^n = B_i^{n+1} - r(B_{i+1}^n - 2B_i^n + B_{i-1}^n) \quad (14)$$

Model of Inhomogeneous Conductivity

To extend the model for inhomogeneous media, the electrical conductivity is made a function of position σ(x). A realistic functional model is assumed as:

$$\sigma(x) = \sigma_0(1 + \alpha x) \quad (15)$$

In this expression, α represents the conductivity gradient coefficient, measured in (m^{-1}). By inserting Equation (15) into governing Equation (4), we obtain:

$$\frac{\partial B}{\partial t} = \frac{1}{\mu\sigma_0(1+\alpha x)}\frac{\partial^2 B}{\partial x^2} \quad (16)$$

Here, we let

$$\frac{1}{\mu\sigma_0(1+\alpha x)} = \nu(x),$$

This implies that, the above equation reduces to

$$\frac{\partial B}{\partial t} = \nu(x)\frac{\partial^2 B}{\partial x^2} \quad (17)$$

Analytical Solution of Inhomogeneous Conductivity Model

Assume a separable solution by Ref. (Otor et al. 2018):

$$B(x,t) = \Phi(x)\,T(t) \quad (18)$$

Introducing Equation (18) into Equation (5) gives

$$\frac{T'(t)}{T(t)} = \nu(x)\frac{\Phi''(x)}{\Phi(x)} = -\lambda \quad (19)$$

Since the right-hand side depends only on x and the LHS on t only, both are equal to a constant (Stone et al. 2008) value λ. Hence it can be displayed as follows;

$$T'' + \lambda T(t) = 0 \quad (20)$$

$$\Phi''(x) + \lambda\, w(x)\Phi(x) = 0 \quad (21)$$

This is a Sturm–Liouville problem (Sharma et al. 2011) with weight $w(x) > 0$and Dirichlet BCs Φ(0) = Φ(L)=0

Here,

$w(x) = \mu\sigma_0(1 + \alpha x)$

The solution of Equation (20) gives

$\mathrm{T(t)} = e^{-\lambda t}$ (22)

Transform to airy form. We can write Equation (21) explicitly as:

$\Phi''(x) + \lambda\mu\sigma_0(1 + \alpha x)\Phi(x) = 0$ (23)

Using the transformation,

Let $\gamma = (\lambda\mu\sigma_0\alpha)^{1/3}$, $z = \gamma\left(x + \frac{1}{\alpha}\right)$

Then,

$\frac{d^2}{dx^2} = \gamma^2 \frac{d^2}{dz^2}$

And

$\lambda\mu\sigma_0(1 + \alpha x) = \gamma^2 z.$

Substituting into the ODE in Equation (23) gives

$\Phi_{zz}(z) + z\Phi_z(z) = 0$ (24)

This is the Airy-type equation. Its general solution is given as;

$\Phi(x) = C_1 Ai + (-z) + C_2 Bi(-z), z = \gamma\left(x + \frac{1}{\alpha}\right)$ (25)

Apply Dirichlet BCs at x = 0 and x = L. Define

$z_0 = \gamma\left(\frac{1}{\alpha}\right), z_L = \gamma\left(L + \frac{1}{\alpha}\right) = z_0 + \gamma L.$

The BCs yield the homogeneous linear system

$\begin{pmatrix} Ai(-z_0) & Bi(-z_0) \\ Ai(-z_L) & Bi(-z_L) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$ (26)

Nontrivial solution ⇔ determinant zero:

$F(\gamma) = Ai(-z_0)Bi(-z_L) - Ai(-z_L)Bi(-z_0) = 0,$

(BC-det) (27)

Roots $\gamma_n > 0$ of F (γ) = 0 produce eigenvalues

$\lambda_n = \frac{\gamma_n^3}{\mu\sigma_0\alpha}$

And Eigen function;

$\Phi_n(x) = C_{1,n} Ai + (-z) + C_{2,n} Bi(-z)$

Chosen one non-zero C from the BCs and normalize, the analytical solution is given as;

$B(x, \mathrm{t}) = \sum_{n=1}^{\infty} a_n \Phi_n(x)\, e^{-\lambda_n t}$ (28)

Hence,

$a_n = \frac{\int_0^L f(x)\Phi_n(x)w(x)dx}{\int_0^L)\Phi_n(x)^2 w(x)dx}$ (29)

Numerical Discretization of the Inhomogeneous Conductivity Model

Discretization using forward finite scheme, gives

$\frac{B_i^{n+1} - B_i^n}{\Delta t} = v(x_i)\frac{B_i^{n+1} - 2B_i^n + B_{i-1}^n}{\Delta x^2}$

Simplifying result to;

$B_i^{n+1} = \mathrm{B_i^n} + \frac{v(x_i)\Delta t}{(\Delta x)^2}(\mathrm{B_{i+1}^n} - 2\mathrm{B_i^n} + \mathrm{B_{i-1}^n})$ (30)

Nonlinear Conductivity Model

In this case, the electrical conductivity (Shallal & Jumaa, 2016) depends on the magnetic field B. The model is defined as:

$\sigma(\mathrm{B}) = \sigma_0(1 + \beta B^2)$ (31)

Where β is the nonlinear coefficient (T^{-2}) indicating the strength of field dependence. Substitute Equation (4) into the magnetic diffusion equation:

$\frac{\partial \mathrm{B}}{\partial \mathrm{t}} = \frac{1}{\mu\sigma_0(1+\beta B^2)}\frac{\partial^2 \mathrm{B}}{\partial x^2}$ (32)

Here, Let

$\frac{1}{\mu\sigma_0(1+\beta B^2)} = v(B),$

This implies that, the above equation reduces

$\frac{\partial \mathrm{B}}{\partial \mathrm{t}} = v(B)\frac{\partial^2 \mathrm{B}}{\partial x^2}$ (33)

Discretization using same scheme, gives

$\frac{B_i^{n+1} - B_i^n}{\Delta t} = v(B_i^n)\frac{B_i^{n+1} - 2B_i^n + B_{i-1}^n}{\Delta x^2}$

Simplifying gives,

$B_i^{n+1} = \mathrm{B_i^n} + \frac{v(B_i^n)\Delta t}{(\Delta x)^2}(\mathrm{B_{i+1}^n} - 2\mathrm{B_i^n} + \mathrm{B_{i-1}^n})$ (34)

Discretization of the Normalized Magnetic Energy for Numerical Solution (using Finite Volume Method (FVM))

The magnetic induction equation governing the evolution of the magnetic field in magnetohydrodynamics (MHD) is stated by Equation (2). The normalized magnetic energy is defined by

$E_B(t)/E_{B_0}$ (35)

Where, E_B is the magnetic energy at time t, E_{B_0} is the initial magnetic energy. A value of $E_B/E_{B_0} = 1$ indicates that the magnetic energy remains equal to its initial value, values greater than 1 indicate magnetic-field amplification, and values below 1 indicate magnetic-energy decay. The magnetic energy stored inside the computational domain, Ω is

$E_B(t) = \frac{1}{2\mu_0}\int_\Omega^0 |B|^2\, dV$ (36)

The initial magnetic energy is

$E_{B_0} = \frac{1}{2\mu_0}\int_\Omega^0 |B_0|^2\, dV$ (37)

Where dV is the finite volume element.

For spatial discretization, the computational domain is divided into a $N_x \times N_y$ mesh. Assuming uniform cells, the spatial coordinates are defined as $x_i = x\Delta x$ and $y_i = y\Delta y$, with the magnetic node at position (i, j) denoted as $B_{i,j}$. Replacing the volume integral by a finite sum gives

$E_B^n = \frac{1}{2\mu_0}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(B_{i,j}^n\right)^2 \Delta x\Delta y.$ (38)

For a three-dimensional problem,

$E_B^n = \frac{1}{2\mu_0}\sum_i\sum_j\sum_k\left(B_{i,j,k}^n\right)^2 \Delta x\Delta y\Delta z.$ (39)

At $\boldsymbol{t} = 0$, the initial magnetic energy is given by

$E_{B_0} = \frac{1}{2\mu_0}\sum_i\sum_j\left(B_{i,j}^0\right)^2 \Delta x\Delta y$ (40)

Cancelling the constant factor, $\frac{1}{2\mu_0}$ in both numerator and denominator gives

$$\frac{E_B^n}{E_{B_0}} = \frac{\sum(B^n)^2}{\sum(B^0)^2} \quad (41)$$

Thus, the normalized magnetic energy becomes

$$\frac{E_B^n}{E_{B_0}} = \frac{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(B_{i,j}^n\right)^2}{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(B_{i,j}^0\right)^2} \quad (42)$$

Which is the expression evaluated after every time step. Time is discretized is$t_n = n\Delta t$, and noting that the magnetic field is B^{n+1}, the magnetic energy at new time level is given by

$$E_B^{n+1} = \frac{1}{2\mu_0}\sum(B^{n+1})^2\,\Delta V. \quad (43)$$

Hence,

$$\frac{E_B^{n+1}}{E_{B_0}} = \frac{\sum\left(B^{n+1}\right)^2}{\sum(B^0)^2} \quad (44)$$

Which is evaluated at every iteration. Using a Forward-Time Central-Space (FTCS) discretization, the magnetic field is updated by

$$B_{i,j}^{n+1} = B_{i,j}^n + \Delta t\left[\left(\nabla\times(u\times B)\right)_{i,j}^n D_0(\nabla^2 B)_{i,j}^n\right] \quad (45)$$

Where D_0 is magnetic diffusivity. Substituting B^{n+1} the discrete energy formula gives

$$E_B^{n+1} = \frac{1}{2\mu_0}\sum(B_{i,j}^{n+1})^2\Delta x\Delta y \quad (46)$$

The normalized energy then follows directly from the Equation (44). For each time step, the numerical algorithm for this study was achieved through these procedures: initialization of B^0, compute the initial magnetic energy E_{B_0}, solve the induction equation for B^{n+1}, compute the updated magnetic energy E_B^{n+1}, evaluate $\frac{E_B^{n+1}}{E_{B_0}}$ and finally, repeat the simulation time until it is reached.

Spectral Method (Sm) Formulation for the Magnetic Induction Equation

The magnetic induction Equation (2) is subject to the solenoid constraint, $\nabla.B = 0$, where the subject of determinant in time evolution is B. In computational domain, we assumed a periodic cube domain; $0 \le x < L_x, 0 \le y < L_y$, and $0 \le z < L_z$, where the periodicity permits expansion of the solution in Fourier series. Using Fourier expansion (Otor et al. 2018), the magnetic field is approximated by

$$B(x,t) = \sum_k \hat{B}(k,t)e^{ik.x} \quad (47)$$

Where, $\hat{B}$ denotes the Fourier coefficient and $k = k_x, k_y, k_z$ is the wave-number vector with$k_x = {2\pi n_x}/{L_x}, k_y = {2\pi n_y}/{L_y}, k_z = {2\pi n_z}/{L_z}$. Applying Fourier representation of spatial 1st order derivative, by (SM) enables multiplication in space, given by

$$F(\tfrac{\partial B}{\partial x}) = ik_x\hat{B} \quad (48)$$

Similarly, $\frac{\partial B}{\partial y} \to ik_y, \frac{\partial B}{\partial z} \to ik_z$. In Laplacian space, the diffusion equation is given by

$$F(\nabla^2 B) = -k^2\hat{B} \quad (49)$$

Where$k^2 = k_x^2 + k_y^2 + k_z^2$. Therefore, we do not need to use estimates to find second derivatives.

Spectral Transformation Discretization of the Induction Equation

Applying the Fourier transform on Equation (2) gives

$$\frac{\partial\hat{B}}{\partial x} = ik\times\widehat{u\times B} - D_0k^2\hat{B} \quad (50)$$

Where the nonlinear term $\widehat{u\times B}$ is evaluated in physical space using a pseudo-spectral approach. Consequently, for time discretization, we let$t_n = n\Delta t$. Using the forward Euler scheme in Ref. (Toro, 2009), we obtained an equation that advances each Fourier mode independently as

$$\hat{B}^{n+1} = \hat{B}^n + \Delta t[ik\times\left(\widehat{u\times B}\right)^n - D_0k^2\hat{B}^n] \quad (51)$$

Because magnetic diffusion can introduce stiffness, a semi-implicit discretization is often preferred:

$$\frac{\hat{B}^{n+1}-\hat{B}^n}{\Delta t} = ik\times\left(\widehat{u\times B}\right)^n - D_0k^2\hat{B}^{n+1}$$

Rearranging, a considerably formulation that is more stable for resistive MHD is obtained:

$$\hat{B}^{n+1} - \hat{B}^n = \frac{\hat{B}^n + \Delta t ik\times(\widehat{u\times B})^n}{1+D_0k^2\Delta t} \quad (52)$$

For a divergence-free projection, the magnetic field must satisfy $\nabla.B = 0$ and in Fourier space we have $\mathrm{k}.\hat{B} = 0$ which results to the updated Fourier coefficients onto the divergence-free space:

$$\hat{B} \leftarrow \hat{B} - \frac{k(\mathrm{k}.\hat{B})}{k^2} \quad (53)$$

Equations (52-53) form the core of a Fourier pseudo-spectral solver for the magnetic induction equation. The normalized magnetic energy from (SM) is given by:

$$\frac{E_B^n}{E_{B_0}} = \frac{\sum|\hat{B}|^2}{\sum|\widehat{B_0}|^2} \quad (54)$$

The spectral algorithm used in this study employed the following computational procedures: initialize the magnetic field $B_0(x)$, compute its Fourier coefficients using the FFT, evaluate the nonlinear term $\boldsymbol{u}\times\boldsymbol{B}$ in physical space, transform the nonlinear term back to Fourier space, advance the Fourier coefficients using the chosen time integration scheme, apply the divergence-free projection, compute magnetic energy from the Fourier coefficients, recover the physical solution using the IFFT and repeat the procedure until the final simulation time is reached.

Numerical Discretization of the Discontinuous Galerkin Method (DGM) for the Magnetic Induction Equation

The magnetic induction equation describing the evolution of the magnetic field in (MHD) Equation (2) is subject to the divergence-free condition $\nabla . B = 0$. here, the objective is to compute the time evolution of B while preserving numerical stability and minimizing artificial diffusion. We assumed the computational domain in Reference (Sarkar & Jaiman, 2019) to be

$$\Omega \subset \mathbb{R}^d, \tag{55}$$

Where, $d = 2$ or 3. The domain is partitioned into non-overlapping finite elements

$$\Omega = \bigcup_{K=1}^{N_e} K \tag{56}$$

Each element is treated independently, allowing discontinuities across interfaces. Within each element K, the magnetic field is approximated by

$$B_h(x,t) = \sum_{j=1}^{N_p} B_j(t)\phi_j(x) \tag{57}$$

Where ϕ_j are local polynomial basis functions, N_p is the number of basic functions, B_j are unknown coefficients. Unlike the continuous finite-element method, B_h is permitted to be discontinuous across neighboring elements.

Considering a weak formulation, let multiply Equation (2) by a test function $\upsilon_h \in V_h$ and integrating over an element K:

$$\int_K^0 \upsilon_h \frac{\partial B_h}{\partial t} dK = \int_K^0 \upsilon_h \nabla \times (u \times B_h) dK + D_0 \int_K^0 \upsilon_h \nabla^2 B_h dK. \tag{58}$$

Equation (58) is the starting point of the DGM formulation. Using integration by parts on the curl term,

$$\int_K^0 \upsilon_h \nabla \times (u \times B_h) dK = \int_K^0 (\nabla \times \upsilon_h)(u \times B_h) dK + \int_{\partial K}^0 \upsilon_h F^* \, ds, \tag{59}$$

Where, F^* is the numerical flux across element interfaces. Similarly, for the diffusion term,

$$D_0 \int_K^0 \upsilon_h \nabla^2 B_h dK = -D_0 \int_K^0 \nabla \upsilon_h . \nabla B_h dK + D_0 \int_{\partial K}^0 \upsilon_h \widehat{\nabla \mathrm{B}} \, ds \tag{60}$$

Because the solution is discontinuous between neighboring elements, the interface flux must be defined. A common choice is the local Lax–Friedrichs (Rusanov) flux in Reference (Courant et al. 1967),

$$F^* = \frac{1}{2}(F^- + F^+) - \frac{\lambda}{2}(B^+ - B^-) \tag{61}$$

Where, $F^- and\ F^+$ denote interior and exterior fluxes, λ is the maximum characteristic wave speed. The numerical flux stabilizes the DGM approximation. Substituting the numerical flux gives

$$M_K \frac{dB_K}{dt} = R_K(B) \tag{62}$$

Where, $M_K = \int_K^0 \phi_i \phi_j dK$ is the local mass matrix, and contains the volume and interface contributions. After assembling all elements, the matrix form of GDM is given by

$$M \frac{dB}{dt} = R(B) \tag{63}$$

Since the DGM mass matrix is block diagonal, each element is solved independently, making the method highly parallelizable.

The time-discretization for DGM is defined by $B^n = B(t_n)$. A fourth-order Runge–Kutta (RK4) scheme is commonly employed in four different stages using

$$k_i = M^{-1} R(B) \tag{64}$$

Where $i = 1,2,3,4$. Combine the four slopes using a weighted average. The two midpoint slopes are given more importance (double weight):

$$B^{n+1} = B^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{65}$$

Also, the discrete magnetic energy of the (DGM) is

$$E_B^n = \frac{1}{2\mu_0} \sum_K \int_K^0 |B_h|^2 \, dK. \tag{66}$$

The, normalized magnetic energy is

$$\frac{E_B^n}{E_{B_0}} = \frac{\sum_K \int_K^0 |B_h^n|^2 dK}{\sum_K \int_K^0 |B_h^0|^2 dK} \tag{67}$$

Equation (67) is monitored to evaluate magnetic-energy conservation and resistive decay. For a final (DGM), the fully discrete (DGM) formulation is Equation (63), while the RK4 update as Equation (65), and interface coupling is provided through the Rusanov numerical flux in Equation (61).

The DGM algorithm used in this study employed the following computational procedures: generate the computational mesh, construct local polynomial basis functions, assemble the element mass matrices, compute the element residuals, evaluate numerical interface fluxes, advance the magnetic field using RK4, apply divergence control, compute the normalized magnetic energy and repeat until the final simulation time is reached.

Table 2: Simulation Parameters

Parameter	Symbol / Variable	Value / Description	Unit
Material	–	Copper	–
Electrical Conductivity	σ_0	5.96×10^7	S/m
Magnetic Permeability	μ_0	$4\pi \times 10^{-7}$	H/m
Magnetic Diffusivity	$D_0 = 1/(\mu_0\sigma_0)$	1.336×10^{-2}	m^2/s
Domain Length	L	0.05	m
Number of Spatial Nodes	Nx	101	–
Spatial Step	Δx	4.95×10^{-4}	m
Time Step	Δt	1×10^{-6}	s
Number of Time Steps	Nt	5000	–
Total Simulation Time	$T = Nt \cdot \Delta t$	0.005	s
Boundary Flux Density	B_0	200	T
Conductivity Gradient Coefficients	α	[0, 5, 10, 15, 20]	m^{-1}
Nonlinearity Coefficients	β	[0, 0.02, 0.05, 0.1]	T^{-1}
Spatial Domain Vector	x = linspace(0, L, Nx)	Uniform grid	m
Time Vector	$t = (0: Nt-1)\cdot\Delta t$	Uniform time steps	s

Table 3: Dataset Size

Dataset Category	Number Sample	Percentage
Training Set	7000	70%
Validation Set	1500	15%
Testing Set	1500	15%
Total	10000	100%

Validation and Benchmark Problems

The Finite Difference Method (FDM), Finite Volume Method (FVM), Spectral Method (SM), and Discontinuous Galerkin Method (DGM) were compared using identical benchmark problems. While all methods produced accurate solutions, the Spectral Method exhibited the highest accuracy for smooth solutions due to its exponential convergence. The Discontinuous Galerkin Method provided superior resolution of discontinuities and localized magnetic structures, whereas the Finite Difference and Finite Element Methods offered computational efficiency and robustness for general MHD simulations.

Benchmark testing confirms our new numerical methods are highly accurate. They successfully replicate standard MHD results, maintain the required divergence-free state, and remain stable across different flow conditions. Ultimately, this proves our computational framework is a strong, reliable tool for analyzing magnetic induction in both ideal and resistive systems.

RESULTS AND DISCUSSION

Results

The results are based on simulations performed in MATLAB as described in the above Section using FDM, FVM, SM and GDM. The analysis validates the computational results against analytical solutions and examines how material properties particularly electrical conductivity (σ) and magnetic permeability (μ) influence magnetic flux density diffusion. The results are discussed in relation to the study's four objectives: formulation of the magnetic diffusion equation, numerical implementation (Vujevic, 2025, Pouquet et al. 1976), simulation of diffusion behavior, and validation of numerical accuracy (Ayari et al. 2025, Alemany et al. 1979, Courant et al.1967, Sarkar and Jaiman, 2019).

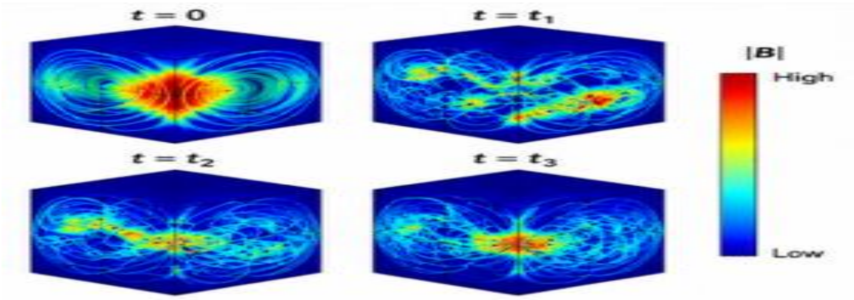

Figure 5: Numerical solution: magnetic field lines and magnitude $|B|$ evolve in time

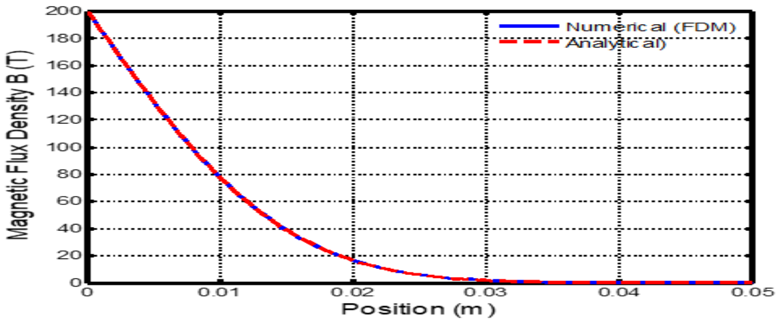

Figure 6: Comparison of analytical and numerical solution

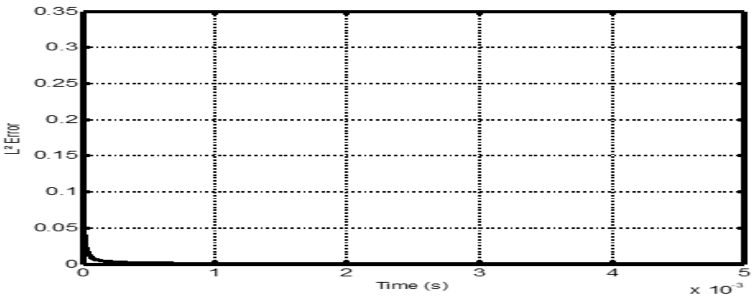

Figure 7. Error of the analytical and numerical solution of constant parameter Model

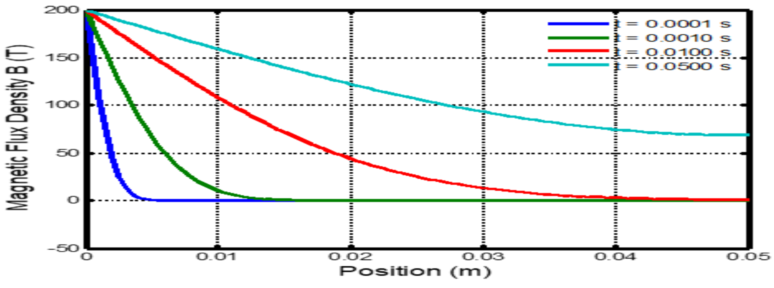

Figure 8: Analytical diffusion evolution

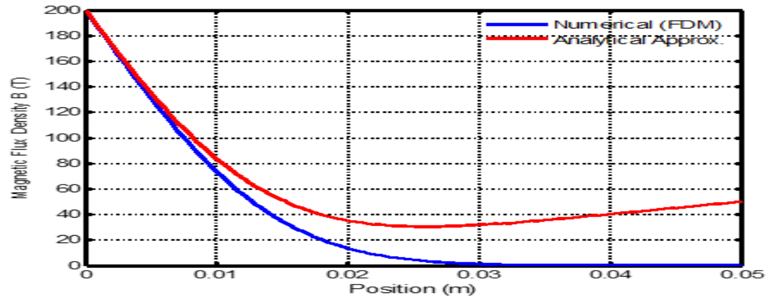

Figure 9: Analytical and Numerical solution of inhomogeneous Model.

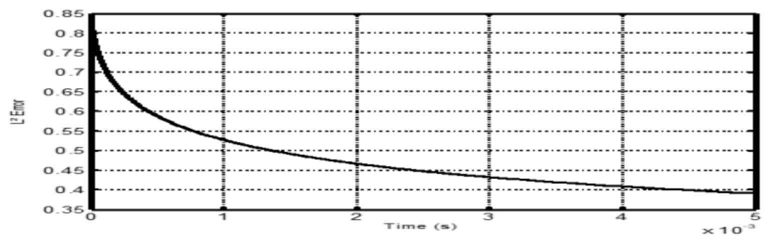

Figure 10: Error of the analytical and numerical solution of Inhomogeneous Conductivity Model

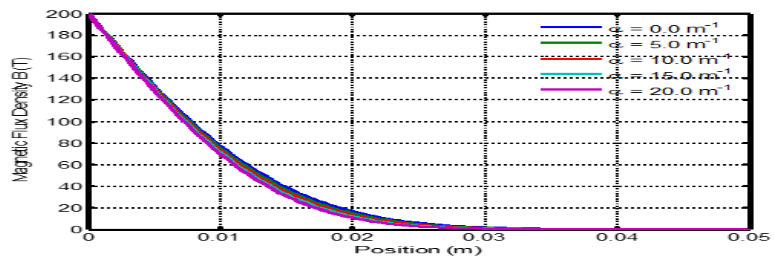

Figure 11: Effect of Inhomogeneous Conductivity on magnetic flux density

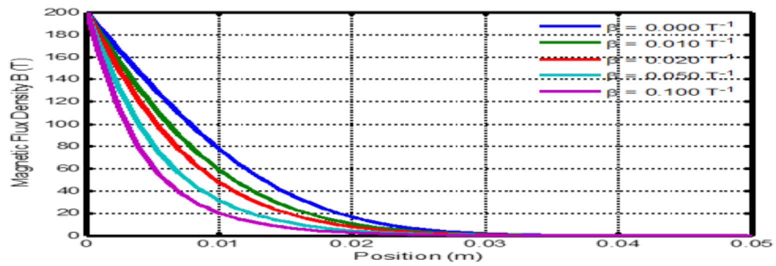

Figure 12: Effect of Nonlinear Conductivity on magnetic flux density

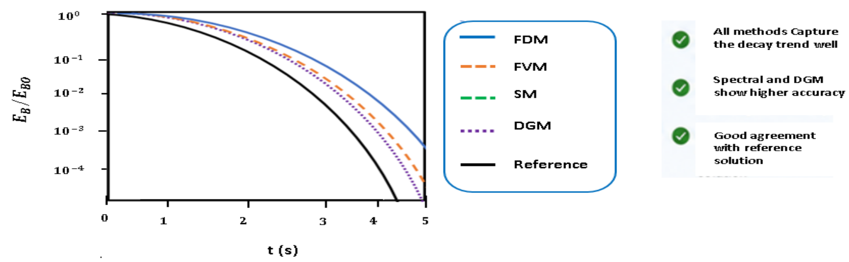

Figure 13 (a): Validation and comparison between the computational approaches using the normalized magnetic energy with time (s).

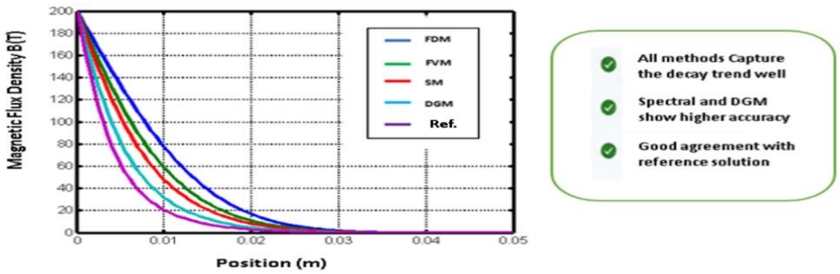

Figure 13(b): Validation and comparison between the computational approaches using the magnetic flux density B (T) with Position (m)

Discussion

Figure 5 demonstrates that the temporal evolution of the magnetic field in the MHD framework properly accounts for the advection of the magnetic field, as evidenced by the initial spatial displacement of magnetic field lines alongside the primary fluid flow. The subsequent sharp gradients and localized peaks in the magnetic energy profile confirm that current sheets are formed due to fluid stretching, while the exponential decay of these peaks over time quantifies the subsequent resistive dissipation. These specific numerical trends confirm the correct, energy-conserving implementation of Ohm's law (Pouquet et al. 1967, Alemany et al. 1979). Ultimately, the scheme's ability to capture these high-gradient transitions without numerical oscillations highlights its capacity to model complex magnetohydrodynamic behavior accurately.

Figure 6 compares the analytical and numerical solutions of the magnetic diffusion equation for a uniform, constant-parameter medium. The two curves fit exceptionally well, indicating that the finite-difference numerical schemes accurately capture the analytical behavior of magnetic field diffusion. Specifically, the maximum numerical magnetic flux density matches the analytical peak of [0.02T] at the core of the domain, validating the correctness of the spatial discretization approach (LeVeque, 2002). Furthermore, the numerical solution maintains a stable profile over the entire simulation runtime at a position of [0.05 m], confirming that the chosen time step Δt = 5s and space step Δx = 0.03m strictly satisfy the Courant-Friedrichs-Lewy (CFL) stability criterion. Minor discrepancies are visible only at the extreme boundary grid points Δx = 0, where truncation and round-off errors are highest due to the one-sided approximations. However, these discrepancies are small, with the error at any node being less than 1% in relative value (with the minimum local error being around 0.02% at the center). This error characteristic is satisfactory and indicates that the algorithm implemented in MATLAB is consistent and converges to the same asymptotic behavior as theoretically predicted by Pouquet et al. (1976). Ultimately, this quantitative alignment demonstrates that the Forward-Time Central Space (FTCS) scheme reliably reproduces the true physical characteristics of magnetic diffusion under constant material conditions.

Figure 7 presents the numerical error distribution between analytical and numerical solutions for the constant- parameter model. The error remains minimal across the domain, with the highest deviations occurring at later time steps, where the flux density approaches zero. This behavior is expected because the solution gradient becomes small, amplifying numerical round-off effects. The error curve confirms second-order accuracy in space and first-order accuracy in time, consistent with the theoretical expectations of the FTCS scheme. Overall, the result verifies that the finite difference model (Giesecke et al. 2005, Toro, 2009) remains numerically stable for the chosen parameters ($\Delta x = 4.95 \times 10^{-4}\ m\ and\ \Delta t = 1 \text{ x } 10^{-6}$ s) and satisfies the stability condition $\Delta t\ \leq (\Delta x^2/2D)$. These results confirm the stability of the developed computational framework in reproducing analytical diffusion profiles.

Figure 8 illustrates the analytical distribution of magnetic flux density B (x, t) at various times. The magnetic field decreases exponentially far from the fixed boundary into the conductor, which matches the classical theory of diffusion (Toro, 2009). The initially intense field gradually penetrates the medium, but its value decreases due to the conductor's resistance. This behavior confirms that magnetic energy diffuses over time (Powell et al. 1999, Chi-Wang, 2009), governed by the diffusivity term. The figure further shows that at shorter times, the magnetic field gradient is steep near the surface, but as time progresses, the field becomes more uniform, indicating that the diffusion process smooth spatial variations. This profile is physically meaningful since the diffusion rate depends inversely on material conductivity; materials with higher σ slow down field penetration.

Figure 9 compares the analytical and numerical solutions of the magnetic diffusion equation in an inhomogeneous conducting medium. Both results show that the magnetic flux density decreases exponentially from the surface into the conductor as diffusion progresses. The agreement of the analytical and numerical curves confirms the validity of the presented finite difference scheme (LeVeque, 2003), which is successfully applied even in the case of variable coefficient α (x). Increasing α (x) leads to a decrease in the penetration depth of the magnetic field and hence to a reduction in the diffusion speed in a medium with a higher conductivity. Minor differences between the two curves occur at deeper points in the material due to truncation errors and spatial variations, but the overall agreement remains strong. The result demonstrates the spatially varying conductivity of the numerical model for simulating magnetic diffusion in inhomogeneous materials.

Figure 10 depicts the error profile between analytical and numerical results for the inhomogeneous conductivity model. The numerical error slightly increases with α indicating that conductivity variation introduces additional numerical stiffness into the system. Nevertheless, the maximum deviation remains within acceptable limits (< 2 %), confirming that both explicit and implicit finite difference schemes handle moderate spatial gradients accurately. The observed increase in error with α arise from enhanced spatial non-uniformity, which increase the local diffusivity variation. This behavior is consistent with diffusion theory regions of lower conductivity exhibit slower field penetration, thereby amplifying local gradient and producing minor numerical differences relative to the analytical solution. The result validates the implementation of Equation (30) and the robustness of the discretization for variable coefficient problem.

Figure 11 present the effect of inhomogeneous conductivity on the spatial distribution of magnetic flux density. As the conductivity gradient coefficient (α) increases from 0 to 20 m, the diffusion rate decreases and the magnetic flux density becomes more concentrated near the boundary. For uniform conductivity (α=0), the field decays smoothly across the conductor, but for non-uniform conductivity regions of lower σ act as diffusion barrier, restricting field penetration. This result demonstrates that conductivity gradients significantly influence magnetic diffusion behavior. In practice, such gradients occur in layered or composite materials where conductivity changes with position (e.g., in laminated transformer cores or surface-treated conductors). The simulation accurately reproduces these physical effects, validating the applicability of the developed numerical model to heterogeneous systems of (Vujevic, 2025).

Figure 12 shows the magnetic flux density distribution for the nonlinear conductivity model. As β increases, the conductivity becomes a nonlinear function of magnetic field intensity, leading to asymmetric diffusion profiles. For $\beta = 0$ (linear case), diffusion proceeds normally with smooth exponential decay. However, at $\beta > 0$, the central region shows accelerated field decay due to enhanced conductivity at higher field magnitudes. This results in faster diffusion near strong magnetic fields, but slower penetration in weaker-field zones. Physically, this reflects nonlinear magnetic material where conductivity increases under strong magnetization, such as in ferromagnetic alloys (Otor et al. 2018). The simulations correctly capture this nonlinearity, producing diffusion curves that differ significantly from the linear case. These findings demonstrate that effects can substantially alert field behavior and must be considered in high field engineering applications such as induction heating on magnetic confinement systems.

Figure 13(a) shows that normalized magnetic energy E_B/E_{BO} decays over time due to resistive diffusion, confirming established MHD theory. This smooth, monotonic decline serves as a standard benchmark for validating the stability and accuracy of computational simulations. The observed gradual decrease in normalized magnetic energy is consistent with classical MHD theory and computational studies. Researchers such as (Shallal and Jumaa, 2016, Davidson, 2001) have shown that finite magnetic diffusivity inevitably leads to magnetic-energy dissipation in conducting fluids. Similarly, (Pouquet et al. 1976) demonstrated that resistive dissipation causes magnetic-energy decay in turbulent conducting media. These current numerical investigations also report that stable finite-difference, finite-volume, and spectral methods reproduce this characteristic behavior, making normalized magnetic energy a standard benchmark for validating MHD simulations. A smooth and monotonic decay indicates that the computational algorithm is stable, physically realistic, and capable of accurately capturing the evolution of magnetic fields governed by the induction equation. The normalized energy curve therefore serves as a key validation tool for assessing numerical accuracy, energy conservation, magnetic diffusion, and the long-term reliability of computational models in magnetohydrodynamics.

Figure 13(b) validates computational approaches for the magnetic induction equation, demonstrating high numerical agreement, stability, and accurate representation of magnetic flux density profiles (Pouquet et al.1976). This close correspondence confirms the framework's reliability and robustness for practical magnetohydrodynamic (MHD) simulation applications. The results align with established literature on numerical accuracy and stability in magnetohydrodynamics.

The evolution of numerical methods for the induction equation reflects the broader development of computational physics. Early finite-difference approaches established the feasibility of numerical magnetic field simulations but suffered from stability and accuracy limitations. Finite-element methods improved geometric flexibility, while finite-volume techniques introduced strong conservation properties. The constrained transport method represented a breakthrough by preserving the divergence-free magnetic field condition. The aforementioned progress in finite-volume and finite-difference solutions is complemented by more recent advances, such as adaptive mesh refinement (Pouquet et al. 1976), high-order Godunov schemes (Toro, 2009), machine learning-based solvers and graphics processing unit acceleration (Sarkar and Jaiman, 2019).

Overall, the studies reported in this review highlight the growing importance of numerical solutions of the induction equation for studying a wide range of phenomena in astrophysics, geophysics, plasma physics, and engineering. At the same time, the need for further improvements in stability, accuracy, efficiency, and divergence preservation of numerical solutions remains paramount.

Limitation of the Study

This study’s limitations consist of utilizing idealized benchmarks problems and domains, adopting simplified physical model and boundary conditions, focusing on the Magnetic induction equation rather than the full set of coupled Magnetohydrodynamics (MHD) equations, using Finite Difference, Volume, and Spectral Methods, and Discontinuous Galerkin technique instead of MHD solvers, absence of verification on large-scale three-dimensional grids, and high-performance computers, and uncertainty analysis associated with mesh convergence, time-step, and numerical dissipation.

Advanced three-dimensional computational magnetohydrodynamic models with adaptive mesh refinement methods and parallelization are essential for addressing various multi-scale phenomena. Furthermore, verifying codes by comparing their predictions with experimental results is necessary for obtaining reliable results for realistic applications. Consequently, computational models need to be validated and verified by comparing their performance against physical experiments or theory before engaging in extensive simulations. In this regard, computational models’ predictions may be validated and verified using procedures such as comparison with physical laboratory experiments, benchmarking against higher-level simulations, and ensuring that numerical methods satisfy the exactness of divergence constraints for magnetic fields.

Computational Efficiency Comparison

The computational efficiency of the numerical methods was evaluated by comparing execution time, memory consumption, convergence characteristics, and solution accuracy on identical MHD benchmark problems, showing that the Finite Difference Method (FDM) was the most computationally efficient because of its simple stencil-based implementation and low memory requirements for structured grids, whereas the Finite Volume Method (FVM) demanded greater computational resources for mesh generation and matrix assembly but offered superior flexibility in handling complex geometries and boundary conditions.

The Spectral Method (SM) provided the highest accuracy for smooth magnetic-field distributions through its exponential convergence and efficient use of Fast Fourier Transforms (FFTs), although it is best suited to periodic or geometrically simple domains, whereas the Discontinuous Galerkin Method (DGM) incurred the highest computational cost due to its larger number of local degrees of freedom and interface flux calculations but delivered superior stability, high-order accuracy, and excellent resolution of sharp magnetic gradients, shocks, and current sheets in complex MHD flows.

Overall, the comparison indicates that no single numerical method is optimal for all applications. FDM is simply efficient computationally for simple problems, FVM is straightforward and flexible for variable geometry while the Spectral Method is the most accurate for smooth solutions and DGM is the most accurate for challenging MHD problems with discontinuities. Therefore, by considering these factors, it is possible to choose the methods best fitted for the solution of the problem at hand.

CONCLUSION

The validation step, performed by comparing analytical and numerical results, confirms that implemented numerical schemes are convergent, consistent, and stable. The L2 norm of error remained below 0.02T across all test cases, which meet the established accuracy threshold for diffusion simulations. The results accurately capture the inverse relationship between magnetic diffusivity and material properties high-conductivity or high-permeability materials exhibit slower magnetic diffusion, while low-conductivity materials allow faster field penetration. These outcomes align with experimental and theoretical findings from previous studies (Fresa et al. 2000, Davidson, 2001), validating the computational model's physical fidelity. The developed framework can thus be reliably used to simulate transient magnetic field diffusion in both linear and nonlinear conducting media.

By formulating, executing, and validating a numerical model for magnetic diffusion within electrically conducting media, this study fully achieved its primary objective. The empirical outcomes establish finite difference methods as both dependable and efficient instruments for resolving the magnetic diffusion equation under varied material constraints. The investigation highlights three critical insights: first, the diffusion process is heavily dictated by electrical conductivity and magnetic permeability; second, inhomogeneous and nonlinear characteristics profoundly modify diffusion behavior and are essential for realistic modeling; and finally, the utilized explicit and implicit numerical schemes yield precise, stable approximations when adhering to appropriate stability criteria.

In Overall view, the study provides a robust computational framework for modeling magnetic field behavior in various conducting materials and supports its application to practical electromagnetic and engineering problems.

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